Quantum trajectories: invariant measure uniqueness and mixing

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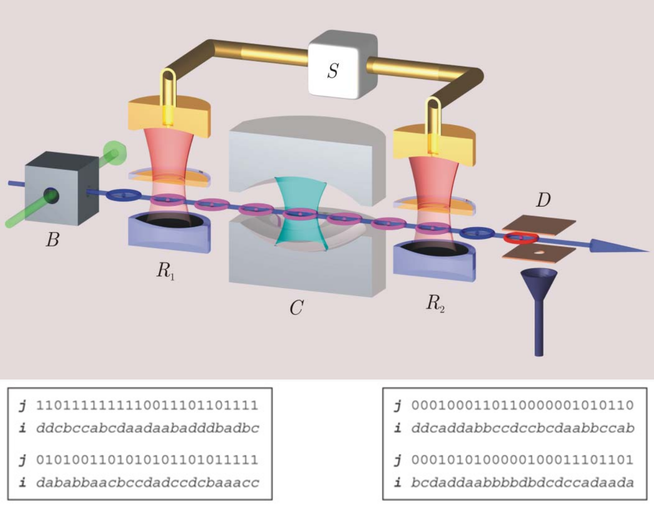

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State space

Definition (Quantum states)

Density matrices :

$$\mathcal{D} := \{\rho \in M_d(\mathbb{C}) \mid \rho \geq 0, \operatorname{tr} \rho = 1\}.$$

Definition (Pure states)

Pure states are the extreme points of $\mathcal{D}$. Namely, $\rho \in \mathcal{D}$ is a pure state iff. $\exists x \in \mathbb{C}^d \setminus \{0\}$ s.t.

$$\rho = P_x := |x\rangle\langle x|.$$

Definition (Metric)

Unitary invariant norm distance:

$$\mathrm{d}(\rho, \sigma) = \|\rho - \sigma\|.$$

Remark

For $U \in U(d)$, $\quad d(U\rho U^*, U\sigma U^*) = d(\rho, \sigma).$

System evolution without conditioning on measurements

Definition (Completely positive trace preserving (CPTP) maps)
Without conditioning on measurement results the system evolution is given by a CPTP map:

$$\Phi : \mathcal{D} \to \mathcal{D}$$
$$\rho \mapsto \sum_{j=1}^{\ell} V_j \rho V_j^*$$

with **Kraus operators** $V_j \in M_d(\mathbb{C})$ *for all* $j = 1, \ldots, \ell$ *s.t.* $\sum_{j=1}^{\ell} V_j^* V_j = \mathrm{Id}_d$.

Remark
Seeing Φ *as arising from the interaction of the system with an auxiliary system (probe), Kraus operators* $V_j = \langle e_i | U\Psi \rangle := \sum_{j=1}^{\ell} U_{ij} \langle e_j | \Psi \rangle$ *with:*

- *The initial state of the probe* $|\Psi\rangle\langle\Psi|$
- *The system–probe interaction* U
- *The observable measured on the probe* $J := \sum_{j=1}^{\ell} j |e_j\rangle\langle e_j|$

Different observables on the probe give different V_j *but same* Φ.

$$\Phi(\rho) := \mathrm{tr}_{probe}(U \rho \otimes P_\Psi U^*) = \sum_{j=1}^{\ell} \langle e_j | U\Psi \rangle \rho \langle \Psi | U^* e_j \rangle.$$

Indirect measurement

Initial state: $\rho \in \mathcal{D}$

- Evolution unconditioned on the measurement: $\rho \mapsto \Phi(\rho)$.

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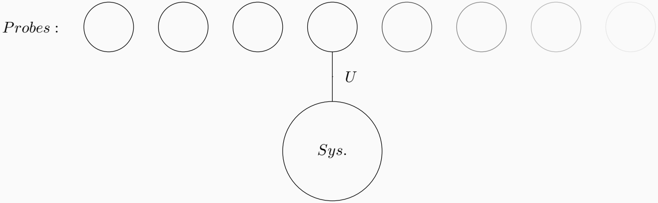
Remark that:

$$\mathbb{E}(\rho' | \rho) = \Phi(\rho).$$

Repeated interactions



- Without conditioning on the measurement, after n interactions: $\bar{\rho}_n = \Phi^{\circ n}(\rho)$.

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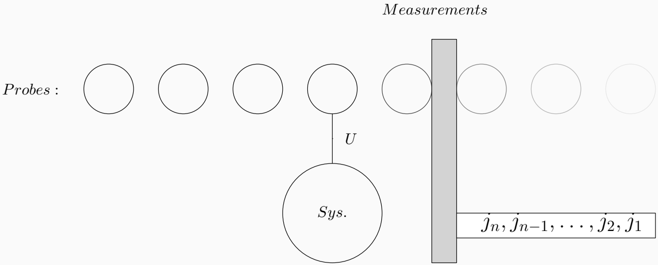

- Without conditioning on the measurement, after n interactions: $\bar{\rho}_n = \Phi^{\circ n}(\rho)$.
- Given the state after $n-1$ measurements of J is ρ_{n-1}, after n measurements of J:

$$\rho_n := \frac{V_j \rho_{n-1} V_j^*}{\mathrm{tr}(V_j^* V_j \rho_{n-1})}, \quad \text{with prob. } \mathrm{tr}(V_j^* V_j \rho_{n-1}).$$

Equivalently, given $\rho_0 = \rho$, after n measurements of J producing result sequence $j_1, \ldots, j_n$:

$$\rho_n := \frac{V_{j_n} \ldots V_{j_1} \rho V_{j_1}^* \ldots V_{j_n}^*}{\mathrm{tr}(V_{j_1}^* \ldots V_{j_n}^* V_{j_n} \ldots V_{j_1} \rho)}, \quad \text{with prob. } \mathrm{tr}(V_{j_1}^* \ldots V_{j_n}^* V_{j_n} \ldots V_{j_1} \rho).$$

Quantum trajectories as Markov chains

Definition (Quantum trajectory)

Given a finite set of $d \times d$ matrices $\{V_j\}_{j=1}^{\ell}$ s.t. $\sum_{j=1}^{\ell} V_j^* V_j = \mathrm{Id}_d$, a quantum trajectory is a realization of the Markov chain of kernel:

$$\Pi(\rho, A) := \sum_{j=1}^{\ell} \mathbf{1}_A \left(\frac{V_j \rho V_j^*}{\mathrm{tr}(V_j^* V_j \rho)} \right) \mathrm{tr}(V_j^* V_j \rho)$$

for any $A \subset \mathcal{D}$ mesurable.

Definition (Irreducibility)

The CPTP map Φ is said irreducible if the only non null orthogonal projector P such that $\Phi(PM_d(\mathbb{C})P) \subset PM_d(\mathbb{C})P$ is $P = \mathrm{Id}_d$.

Theorem (Evans, Høegh-Krohn '78)

A CPTP map $\Phi : M_d(\mathbb{C}) \to M_d(\mathbb{C})$ is irreducible iff. $\exists ! \rho_{inv.} \in \mathcal{D}$ s.t. $\rho_{inv.} > 0$ and $\Phi(\rho_{inv.}) = \rho_{inv.}$.

Moreover, if Φ is irreducible, its modulus 1 eigenvalues are simple and form a finite sub group of $U(1)$. The sub group size $m \in \{1, \dots, d\}$ is equal to the period of Φ and $\exists\ 0 < \lambda < 1$ and $C > 0$ s.t. $\forall \rho \in \mathcal{D}$,

$$\left\| \frac{1}{m} \sum_{r=0}^{m-1} \Phi^{\circ(mn+r)}(\rho) - \rho_{inv.} \right\| \leq C\lambda^n.$$

Preliminary results: Strong law of large numbers for the state

Theorem (Kümmerer, Maassen '04)
Let $(\rho_n)_n$ be a quantum trajectory. Then,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \rho_k = \rho_\infty \quad a.s.$$

with $\Phi(\rho_\infty) = \rho_\infty$. Particularly, if Φ is irreducible, $\rho_\infty = \rho_{inv}$. a.s.

Preliminary results: Purification

For any $\rho \in \mathcal{D}$ let $S(\rho) := -\operatorname{tr}(\rho \log \rho)$ be its von Neumann entropy. Then:

$$S(\rho) = 0 \iff \operatorname{rank}(\rho) = 1 \iff \rho \text{ is a pure state.}$$

Theorem (Kümmerer, Maassen '06)
The following statements are equivalent:

1. *An orthogonal projector π s.t. $\pi V_{j_1}^* \cdots V_{j_p}^* V_{j_p} \cdots V_{j_1} \pi \propto \pi$ for all $j_1, \ldots j_p \in \{1, \ldots, \ell\}$ is of rank 1,*
2. *For any $\rho_0 \in \mathcal{D}$,*
$$\lim_{n \to \infty} S(\rho_n) = 0 \quad a.s.$$

Remark

- *If π is s.t. $\pi V_{j_1}^* \cdots V_{j_p}^* V_{j_p} \cdots V_{j_1} \pi \propto \pi$ for all $j_1, \ldots j_p \in \{1, \ldots, \ell\}$, there exists unitary matrices $U^{\pi}_{j_1, \ldots, j_p}$ s.t.*
$$V_{j_p} \cdots V_{j_1} \pi \propto U^{\pi}_{j_1, \ldots, j_p} \pi.$$
- *In dimension $d = 2$, either $\lim_{n \to \infty} S(\rho_n) = 0$ a.s., or all the matrices V_j are proportional to unitary matrices.*

Uniqueness and convergence towards the invariant measure

Theorem (B., Fraas, Pautrat, Pellegrini '17)
If the following two assumptions are verified,

(Φ-erg.) Φ *is irreducible,*

(Pur.) *Any orthogonal projector* π *s.t.* $\pi V_{j_1}^* \cdots V_{j_p}^* V_{j_p} \cdots V_{j_1} \pi \propto \pi$ *for all* $j_1, \dots j_p \in \{1, \dots, \ell\}$ *is of rank* 1*,*

Π *accepts a* **unique invariant probability measure** $\nu_{inv.}$.

Moreover, $\exists\ 0 < \lambda < 1$ *and* $C > 0$ *s.t. for any probability measure* ν *over* $\mathcal{D}$*,*

$$W_1\left(\frac{1}{m}\sum_{r=0}^{m-1} \nu\Pi^{mn+r}, \nu_{inv.}\right) \leq C\lambda^n$$

with $m \in \{1, \dots, d\}$ *the period of* Φ.

Previous similar results

- **Products of i.i.d. (Furstenberg, Guivarc'h, Kesten, Le Page, Raugi ... '60–'80, Books: Bougerol et Lacroix '85, Carmona et Lacroix '90)** Markov kernel:

$$\Pi_0(\rho, A) = \sum_{j=1}^{\ell} \mathbf{1}_A \left(\frac{V_j \rho V_j^*}{\mathrm{tr}(V_j^* V_j \rho)} \right) p_j$$

  with $(p_j)_{j=1}^{\ell}$ a probability measure over $\{1, \dots, \ell\}$.
- **Generalization (Guivarc'h, Le Page '01–'16)** Markov kernel:

$$\Pi_s(\rho, A) = \mathcal{N}(s)^{-1} \sum_{j=1}^{\ell} \mathbf{1}_A \left(\frac{V_j \rho V_j^*}{\mathrm{tr}(V_j^* V_j \rho)} \right) \left(\mathrm{tr}(V_j^* V_j \rho) \right)^s p_j$$

  for $s \geq 0$ (Q. Traj.: $s = 1$).

No assumption that $\sum_j V_j^* V_j = \mathrm{Id}_d$ but the matrices V_j need be invertible and a stronger irreducibility condition is assumed.

- $\{V_j\}_{j=1}^{\ell}$ is strongly irreducible(*i.e.* no non trivial finite union of proper subspaces is preserved by the matrices V_j. Then, strong irreducibility $\implies$ **(Φ-erg.)**),
- The smallest closed sub semigroup of $GL_d(\mathbb{C})$ containing $\{V_j\}_{j=1}^{\ell}$ is contracting (equivalent to **(Pur.)** for a strongly irreducible family of invertible matrices).

Examples

- Let $p \in]0,1[$ and

$$V_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & 0 \\ \sqrt{1-p} & 0 \end{pmatrix}, \quad V_3 = \begin{pmatrix} \sqrt{p} & 0 \\ 0 & 0 \end{pmatrix}, \quad V_4 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}.$$

  The family $\{V_1, V_2, V_3, V_4\}$ verifies conditions **(Φ-erg.)** and **(Pur.)**.

- Let,

$$Z = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad X = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

  The family $\{Z, X\}$ verifies **(Φ-erg.)** but not **(Pur.)**. There exists uncountably many mutually singular Π-invariant probability measures concentrated on the pure states.

- Let,

$$Z = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i} & 0 \\ 0 & e^{-i} \end{pmatrix} \quad \text{and} \quad X = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos 1 & i \sin 1 \\ i \sin 1 & \cos 1 \end{pmatrix}.$$

  The family $\{Z, X\}$ verifies **(Φ-erg.)** but not **(Pur.)**. Nevertheless Π accepts a unique invariant probability measure concentrated on pure states.

Non suitable methods

Let, $e_0 = (1,0)^\mathsf{T}$, $e_1 = (0,1)^\mathsf{T}$ and

$$V_1 = \begin{pmatrix} 0 & \sqrt{1-p} \\ \sqrt{p} & 0 \end{pmatrix} \quad \text{and} \quad V_2 = \begin{pmatrix} \sqrt{p} & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}$$

with $p \in]0, 1/2[$. The family $\{V_1, V_2\}$ defines a CPTP map Φ and verifies **(Φ-erg.)** and **(Pur.)**.

ϕ-irreducibility: $\Pi^n(P_{e_{0/1}}, \{P_{e_0}, P_{e_1}\}) = 1$ for any n. Hence, if Π is ϕ-irreducible it is so only for $\phi \ll \frac{1}{2}(\delta_{P_{e_0}} + \delta_{P_{e_1}})$. Though $\Pi^n(P_{e_+}, \cdot\ )$ is atomic and $\Pi^n(P_{e_+}, \{P_{e_0}, P_{e_1}\}) = 0$ for any n with $e_+ = \frac{1}{\sqrt{2}}(1,1)^\mathsf{T}$. Hence

$$\phi(A) > 0 \implies P(\tau_A < \infty | \rho_0 = P_{e_+}) = 0$$

. Particularly,

$$\left\| \delta_{P_{e_0}} \Pi^n - \delta_{P_{e_+}} \Pi^n \right\|_{TV} = 1, \quad \forall n \in \mathbb{N}.$$

Contractivity: For all $n \in \mathbb{N}$ and $j_1, \ldots, j_n \in \{1, \ldots, \ell\}$,

$$d\left(\frac{V_{j_n} \ldots V_{j_1} P_{e_0} V_{j_1}^* \ldots V_{j_n}^*}{\operatorname{tr}(V_{j_1}^* \ldots V_{j_n}^* V_{j_n} \ldots V_{j_1} P_{e_0})}, \frac{V_{j_n} \ldots V_{j_1} P_{e_1} V_{j_1}^* \ldots V_{j_n}^*}{\operatorname{tr}(V_{j_1}^* \ldots V_{j_n}^* V_{j_n} \ldots V_{j_1} P_{e_1})} \right) = 1.$$

Proof of uniqueness structure

1. Assuming **(Φ-erg.)**, for any Π-invariant probability measure, the distribution of the sequences $(j_n)_n$ of J measurement results is the same,
2. Assuming **(Pur.)**, there exists a process $(\sigma_n)_n$ taking value in $\mathcal{D}$ and depending only on $(j_n)_n$ s.t.
$$\lim_{n\to\infty} d(\rho_n, \sigma_n) = 0 \quad a.s.$$

Measurement results unique invariant measure

Lemma

Assume **(Φ-erg.)** *holds. Then, for any* Π*-invariant probability measure* ν *over* $\mathcal{D}$*,*

$$\mathrm{Prob}(j_1, \dots, j_n | \rho_0 \sim \nu) = \mathrm{tr}(V_{j_1}^* \dots V_{j_n}^* V_{j_n} \dots V_{j_1} \rho_{inv.})$$

with $\rho_{inv.}$ *the unique element of* $\mathcal{D}$ *s.t.* $\Phi(\rho_{inv.}) = \rho_{inv.}$*.*

Proof.

Given a fixed initial state, the distribution of J measurement results is given by:

$$\mathrm{Prob}(j_1, \dots, j_n | \rho_0 = \rho) = \mathrm{tr}(V_{j_1}^* \dots V_{j_n}^* V_{j_n} \dots V_{j_1} \rho).$$

Linearity in ρ implies,

$$\mathbb{E}_\nu \left[\mathrm{Prob}(j_1, \dots, j_n | \rho_0 = \rho)\right] = \mathrm{Prob}(j_1, \dots, j_n | \rho_0 = \rho_\nu)$$

with $\rho_\nu = \mathbb{E}_\nu[\rho]$.

Recall that $\mathbb{E}_\nu(\rho_1) = \Phi(\rho_\nu)$, but the Π-invariance of ν implies $\mathbb{E}_\nu(\rho_1) = \rho_\nu$. Hence Perron-Frobenius Theorem of positive linear maps imply ρ_ν is the unique fixed point state of Φ. □

Polar decomposition

Set,

$$W_n := V_{j_n} \dots V_{j_1}.$$

Definition

Let $(M_n)_n$ *be the process:*

$$M_n := \frac{W_n^* W_n}{\mathrm{tr}(W_n^* W_n)} \quad \textit{if } W_n \neq 0$$

and arbitrarily fixed in any other case.

Definition

Let U_n *and* D_n *be two processes s.t.* $U_n D_n = W_n$ *is a polar decomposition of* W_n*.*

Remark

$$\rho_n = \frac{W_n \rho W_n^*}{\mathrm{tr}(W_n^* W_n \rho)} = U_n \frac{\sqrt{M_n}\rho\sqrt{M_n}}{\mathrm{tr}(M_n \rho)} U_n^* \quad a.s.$$

Asymptotic rank one POVM

Proposition

Let $\rho_{ch} := \mathrm{Id}_d / d$.

(i) *For any probability measure* ν *over* $\mathcal{D}$,

$$M_\infty := \lim_{n\to\infty} M_n$$

exists a.s. and in L^1*-norm. Moreover* $\mathbb{E}(M_\infty | \rho_0 = \rho_{ch}) = \rho_{ch}$.

(ii) *The process* M_n *is a positive bounded martingale w.r.t.* $\mathrm{Prob}(\,\cdot\,|\rho_0 = \rho_{ch})$. *It follows that for any* $\rho \in \mathcal{D}$,

$$\mathrm{dProb}(\,\cdot\,|\rho_0 = \rho) = d\,\mathrm{tr}(M_\infty \rho)\,\mathrm{dProb}(\,\cdot\,|\rho_0 = \rho_{ch}).$$

(iii) *If* **(Pur.)** *holds, there exists a random varibale* z *taking value in* $\mathbb{C}^k \setminus \{0\}$ *s.t.*

$$M_\infty = P_z \quad a.s.$$

Remark

- z *depends only on* $(j_n)_n$,
- *The explicit expression of* $\mathrm{dProb}(\,\cdot\,|\rho_0 = \rho)/\,\mathrm{dProb}(\,\cdot\,|\rho_0 = \rho_{ch})$ *implies that,*

$$\mathrm{tr}(\rho P_z) > 0 \quad a.s.$$

Convergence towards a process depending only on the J measurement results

Lemma

Assume **(Pur.)** *holds. Let $(\sigma_n)_n$ be the process taking value in $\mathcal{D}$ defined by,*

$$\sigma_n = U_n P_z U_n^*.$$

Then,

$$\lim_{n\to\infty} d(\rho_n, \sigma_n) = 0 \quad a.s.$$

Proof.

$$\lim_{n\to\infty} U_n^* \rho_n U_n = \lim_{n\to\infty} \frac{\sqrt{M_n}\rho\sqrt{M_n}}{\operatorname{tr}(M_n\rho)} = \frac{P_z\rho P_z}{\operatorname{tr}(P_z\rho)} = P_z \quad a.s.$$

The lemma follow from $\operatorname{tr}(P_z\rho) > 0$ a.s. and

$$d(\rho_n, \sigma_n) = d(U_n^*\rho_n U_n, P_z).$$

□

Uniqueness proof

The uniqueness of the invariant measure follows then from a simple $\epsilon/3$ argument.

Let ν_a and ν_b be two Π-invariant probability measures over $\mathcal{D}$.

Since $(\sigma_n)_n$ depends only on the sequence $(j_n)_n$, the first lemma implies:

$$(\sigma_n)_n \text{ w.r.t. } \nu_a \sim (\sigma_n)_n \text{ w.r.t. } \nu_b$$

Then $\rho_n \sim \nu_{a/b}$ and the a.s. convergence $d(\rho_n, \sigma_n) \to 0$ w.r.t. both $\nu_{a/b}$ implies $\nu_a = \nu_b$.

Convergence

Theorem (BFPP '17)
If assumptions **(Φ-erg.)** *and* **(Pur.)** *hold, then there exists* $0 < \lambda < 1$ *and* $C > 0$ *s.t. for any probability measure* ν *over* $\mathcal{D}$*,*

$$W_1\left(\frac{1}{m}\sum_{r=0}^{m-1} \nu \Pi^{mn+r}, \nu_{inv.}\right) \leq C\lambda^n$$

with $m \in \{1, \ldots, d\}$ *the period of* Φ*.*

Proof structure

The proof is again split in two.

- **(Φ-erg.)** $\implies$ $\left\| \frac{1}{m} \sum_{r=0}^{m-1} \Phi^{\circ mn+r}(\rho) - \rho_{inv.} \right\| \leq C\lambda^n \implies$

$$\left\| \frac{1}{m} \sum_{r=0}^{m-1} \mathrm{Prob}(\,\cdot\,|\Phi^{mn+r}(\rho_0)) - \mathrm{Prob}(\,\cdot\,|\rho_0 = \rho_{inv.}) \right\|_{TV} \leq C\lambda^n.$$

- **(Pur.)** $\implies$ $\exists\ (\hat{\rho}_n)_n$ taking value in $\mathcal{D}$ and depending only on $(j_n)_n$ s.t. for any probability measure ν over $\mathcal{D}$,

$$\mathbb{E}_\nu(d(\rho_n, \hat{\rho}_n)) \leq C\lambda^n.$$

The result follows then from an $\epsilon/3$ argument over the expectation of 1-Lipschitz functions and Kantorovich-Rubinstein duality theorem.

Estimate $\hat{\rho}_n$ definition

Definition

Let $(\hat{P}_n)_n$ be the sequence of maximum likelihood estimates of the quantum trajectory initial state.

$$\hat{P}_n := \operatorname{argmax}_{\rho \in \mathcal{D}} \operatorname{tr}(V_{j_1}^* \dots V_{j_n}^* V_{j_n} \dots V_{j_1} \rho)$$

Proposition

- *The estimate $(\hat{P}_n)_n$ is in general not consistent.*
- *If assumption* **(Pur.)** *holds, then,*

$$\lim_{n \to \infty} \hat{P}_n = P_z \quad a.s.$$

Definition

$$\hat{\rho}_n := \frac{W_n \hat{P}_n W_n^*}{\operatorname{tr}(W_n \hat{P}_n W_n^*)} = U_n \hat{P}_n U_n^*.$$

Lemma

Assume **(Pur.)** *holds. Then there exists $C > 0$ and $0 < \lambda < 1$ s.t. for any probability measure ν,*

$$\mathbb{E}_\nu(d(\rho_n, \hat{\rho}_n)) \leq C \lambda^n.$$

Proof limitations

- The definition of **(Pur.)** is unsatisfactory. It is difficult to check for explicit matrices V_j,
- No information on the continuity of the invariant probability measure.