Quantum Stochastic Thermodynamics: linking some definitions and their properties through modular formalism

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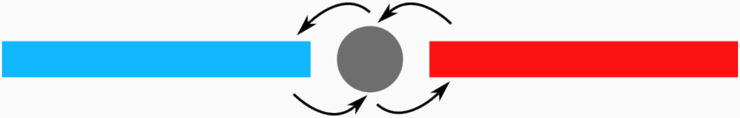
Typical physical setup



Study the fluctuations of heat or entropy production.

$$H = H_c + H_h + V.$$

$$\mathrm{ep} = \frac{1}{T_c} J_{Q_c} + \frac{1}{T_h} J_{Q_h} \geq 0.$$

Quantum stochastic thermodynamics

- Hilbert space: $\mathcal{H}$,
- Dynamics: $(U^t)_t$ unitary group ($U^t = e^{-itH}$),
- Entropy observable: $-\log \rho$ with ρ a reference state

$$\rho \propto \exp(-\beta_c H_c - \beta_h H_h).$$

- Entropy production rate: $\sigma = -i[H, \log \rho]$?
- Entropy production: $\Delta S_t = \int_0^t U^{-s} \sigma U^s ds = -U^{-t} \log \rho U^t + \log \rho$?

Issues:

- No clear physical interpretation of the measurement of ΔS_t,
- Continuous measurement of σ can lead to Zeno effect,
- No fluctuation relation except in few trivial cases.

Two time measurement definition

Difference between two measurements of

$$-\log\rho = \beta_h(H_h - F_h) + \beta_c(H_c - F_c) = \sum_e eP_e.$$

1. Start in a state ν and measure $-\log\rho$ obtaining result e,

$$\nu \rightsquigarrow P_e\nu P_e / \operatorname{tr}(P_e\nu), \quad \text{Probab}(e|\nu) = \operatorname{tr}(P_e\nu),$$

2. Let evolve with U^t,

$$P_e\nu P_e / \operatorname{tr}(P_e\nu) \rightsquigarrow U^t \frac{P_e\nu P_e}{\operatorname{tr}(P_e\nu)} U^{-t},$$

3. Measure $-\log\rho$, obtaining e',

$$\text{Probab}(e'|\nu, e, t) = \operatorname{tr}\big(P_{e'} U^t P_e\nu P_e U^{-t}\big) / \operatorname{tr}(P_e\nu).$$

As a result:

$$\text{Probab}_{\nu,t}(\Delta S_t = s) = \sum_{e'-e=s} \operatorname{tr}\big(P_{e'} U^t P_e \tilde{\nu} U^{-t}\big)$$

with

$$\tilde{\nu} = \sum_e P_e\nu P_e.$$

Two time measurement definition

Definition by Laplace's transform:

$$\mathcal{F}_{\nu,t}(\alpha) = \sum_s e^{-\alpha s}\,\mathrm{Probab}_{\nu,t}(\Delta S_t = s) = \mathrm{tr}\big(\rho_{-t}^{\alpha}\rho^{-\alpha}\tilde{\nu}\big)$$

with $\rho_{-t} = U^{-t}\rho U^{t}$ and

$$\tilde{\nu} = \lim_{R\to\infty}\frac{1}{R}\int_0^R \rho^{i\theta}\nu\rho^{-i\theta}d\theta.$$

$$\mathcal{F}_{\nu,t}(\alpha) = \lim_{R\to\infty}\frac{1}{R}\int_0^R \mathrm{tr}\Big[\nu\rho^{i\theta}\rho_{-t}^{\alpha}\rho^{-\alpha}\rho^{-i\theta}\Big]d\theta.$$

- Clear physical interpretation,
- Automatic fluctuation relation when $\nu = \rho$ (Evans-Searles),
- May makes mathematical sense in the thermodynamic limit,
- What happens when $\nu \neq \rho$, especially for non equilibrium states?

Towards the thermodynamic limit

Timescales: $\tau_{recc.} \gg \tau_{mix.}$, $\tau_{recc.} \gg \tau_{sys.}$. The first limit is a volume one: $L \to \infty$.

Goal: write with objects surviving this thermodynamic limit.

Quasi-local algebra $\mathcal{O}$: $O \in \mathcal{O} \implies \|O - O_L\| \xrightarrow[L\to\infty]{} 0$ with O_L local.

$$O = \sum_{x\in\mathbb{Z}} e^{-|x|}\sigma_3^x \text{ is quasi-local.}$$

$$H = \sum_{x\in\mathbb{Z}} \sigma_3^x\sigma_3^{x+1} \text{ is not quasi-local.}$$

State: $\nu : \mathcal{O} \to \mathbb{C}$, such that $\forall L$ there exists $\hat{\nu}_L$ such that $\nu(O) \simeq \mathrm{tr}(\hat{\nu}_L O_L)$.

Dynamics: $\tau^t : \mathcal{O} \to \mathcal{O}$, $\tau^t(O) \simeq e^{itH_L} O_L e^{-itH_L}$.

Modular dynamic: $\varsigma_\rho^\theta : \mathcal{O} \to \mathcal{O}$, $\varsigma_\rho^\theta(O) \simeq \hat{\rho}_L^{i\theta} O_L \hat{\rho}_L^{-i\theta}$.

Representations

Not sufficient, we want a Hilbert space back.

For $L < \infty$, **thermofield double**:

$$O_L \equiv O_L \otimes \mathrm{Id} =: \pi(O_L)$$

$$\nu_L \equiv (\nu_L^{\frac{1}{2}} \otimes \mathrm{Id})|\Omega\rangle =: |\Omega_\nu\rangle$$

with $|\Omega\rangle = \sum_{j=1}^{d^L} e_j \otimes e_j$.

$$\mathrm{tr}(\nu_L O_L) = \langle \Omega_\nu | \pi(O) | \Omega_\nu \rangle.$$

Hilbert space: $\mathcal{H}_\nu = \overline{\{\pi(O)|\Omega_\nu\rangle : O \in \mathcal{O}\}}$.

Dynamics: $\exists \mathcal{L}$ such that $\pi(\tau^t(O)) = e^{it\mathcal{L}}\pi(O)e^{-it\mathcal{L}}$. In the thermofield double:

$$\mathcal{L} = H \otimes -H.$$

Modular operator: Equivalent to $\nu(x)/\rho(x)$: $\Delta_{\nu|\rho}\pi(O) \simeq \nu O \otimes \rho^{-1}$.

Conne's cocycle : Another equivalent to $\nu(x)/\rho(x)$:

$$[D\nu : D\rho]_\alpha = \Delta_{\nu|\rho}^{\alpha} \Delta_{\rho|\rho}^{-\alpha} \simeq \nu^\alpha \rho^{-\alpha} \otimes \mathrm{Id}\,.$$

Two time measurement definition: modular formalism

The random entropy production is given by the spectral measure of

$$-\log \Delta_{\rho_{-t}|\rho}$$

with respect to $|\Omega_{\tilde{\nu}}\rangle$ with

$$\tilde{\nu}(O) = \lim_{R\to\infty} \frac{1}{R} \int_0^R \nu \circ \varsigma_\rho^\theta(O) d\theta$$

if $\tilde{\nu}$ is a quasi-local perturbation of ρ. Remark: $-\log \Delta_{\rho_{-t}|\rho} \notin \pi(\mathcal{O})$.

Then,

$$\mathcal{F}_{\nu,t}(\alpha) = \langle \Omega_{\tilde{\nu}} | \Delta^{\alpha}_{\rho_{-t}|\rho} | \Omega_{\tilde{\nu}} \rangle.$$

Assuming $[D\rho_{-t} : D\rho]_\alpha \in \pi(\mathcal{O})$,

$$\mathcal{F}_{\nu,t}(\alpha) = \lim_{R\to\infty} \frac{1}{R} \int_0^R \nu \circ \varsigma_\rho^\theta([D\rho_{-t} : D\rho]_\alpha) d\theta.$$

Two time measurement definition: Thermodynamic limit

Assume ν is a thermodynamic limit: $\nu = \lim_{L\to\infty} \nu_L$.

Theorem (B., Bruneau, Jakšić, Panati, Pillet 2024)
Under standard assumptions, for any ν quasi-local perturbation of ρ, $\mathcal{F}_{\nu,t}(\alpha)$ is well defined and

$$\begin{aligned}\lim_{L\to\infty} \mathcal{F}_{\nu_L,t}(\alpha) &= \lim_{L\to\infty}\lim_{R\to\infty} \frac{1}{R}\int_0^R \nu\circ\varsigma_\rho^\theta([D\rho_{-t} : D\rho]_\alpha)d\theta \\ &= \lim_{R\to\infty}\lim_{L\to\infty} \frac{1}{R}\int_0^R \nu\circ\varsigma_\rho^\theta([D\rho_{-t} : D\rho]_\alpha)d\theta \\ &= \mathcal{F}_{\nu,t}(\alpha).\end{aligned}$$

$\mathcal{F}_{\nu,t}$ is extended to any state ν by continuity.

That is a definition of stochastic entropy production with respect to any initial state.

Two time measurement definition: Thermodynamic limit proof

Let $(\mathfrak{H}, \pi, \Omega)$ be the GNS representation of ρ.

- **Modular conjugation** $J\Delta_{\nu|\rho}^{\frac{1}{2}}\pi(A)\Omega_\rho = \pi(A^*)\Omega_\nu$. J is SWAP and complex conjugation in the thermofield double.
- using this representation and von Neumann ergodic theorem,

$$\mathcal{F}_{\nu,t}(\alpha) = \langle J\pi(A^*A)J\Omega, P[D\rho_{-t} : D\rho]_\alpha \Omega \rangle$$

  with P the orthogonal projection onto invariant states under the modular dynamic ς_ρ.

Definition (Ergodic)
The dynamics ς_ρ is ergodic if for any quasi-local perturbation ν of ρ,

$$\lim_{R\to\infty} \frac{1}{R} \int_0^R \nu \circ \varsigma_\rho^\theta(O) d\theta = \rho(O).$$

Super stability of the the two time measurement definition

Theorem (B., Bruneau, Jakšić, Panati, Pillet 2024)
Assume ς_ρ is ergodic, then for any state ν,

$$\mathcal{F}_{\nu,t}(\alpha) = \mathcal{F}_{\rho,t}(\alpha).$$

Assume $\mathcal{O} = \mathcal{O}_S \otimes \mathcal{O}_R$ with $\mathcal{O}_S$ finite dimensional, $\tau_R^t : \mathcal{O}_R \to \mathcal{O}_R$ ergodic and $\rho = \mathrm{Id} / \dim(\mathcal{H}_S) \otimes \rho_R$, then,

$$\mathcal{F}_{\nu,t}(\alpha) = \mathcal{F}_{\nu_S \otimes \rho_R}(\alpha)$$

with $\nu_S = \nu|_{\mathcal{O}_S}$. If $\nu_S > 0$, there exists $C > 0$ such that for α real,

$$C^{-1}\mathcal{F}_{\nu,t}(\alpha) \leq \mathcal{F}_{\rho,t}(\alpha) \leq C\mathcal{F}_{\nu,t}(\alpha).$$

Remark
The decoherence due to the first $-\log\rho$ measurement forces the initial state back to the reference state.

Assumption of perfect measurement implies its timescale is at least similar to the return to equilibrium time scale.

Stability with respect to time

Let α real be such that $\lim_{t\to\infty} \frac{1}{t} \log \mathcal{F}_{\rho,t}(\alpha)$ exists.

Corollary
If $\rho_{T,S} > 0$, then

$$\lim_{t\to\infty} \frac{1}{t} \log \mathcal{F}_{\rho_T,t}(\alpha) = \lim_{t\to\infty} \frac{1}{t} \log \mathcal{F}_{\rho,t}(\alpha).$$

Assume moreover that the non-equilibrium steady state $\rho_+ = \lim_{T\to\infty} \rho_T$ exists.

Corollary
If $\rho_{+,S} > 0$,

$$\lim_{t\to\infty} \lim_{T\to\infty} \frac{1}{t} \log \mathcal{F}_{\rho_T,t}(\alpha) = \lim_{T\to\infty} \lim_{t\to\infty} \frac{1}{t} \log \mathcal{F}_{\rho_T,t}(\alpha).$$

Two-time measurement issues

- The first measurement trivializes the thermodynamic,
- Invasive extended infinitely precise measurement,
- Experimentally inaccessible.

Ancilla tomography computation

Idea: perform a physical Fourier transform. Emerged in physics in 2014 (see De Chiara *et al.* 2018 for a review).

- $\mathcal{H} \rightsquigarrow \mathcal{H} \otimes \mathbb{C}^2$
- $\nu \rightsquigarrow \nu \otimes \rho_a$, where $\rho_a = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$
- $U^t \rightsquigarrow e^{\frac{\alpha}{2}\log\rho\otimes\sigma_z}\left(U^t \otimes \mathbb{1}_{\mathbb{C}^2}\right)e^{-\frac{\alpha}{2}\log\rho\otimes\sigma_z}$,
- $\partial_t \tau_\alpha^t = \tau_\alpha^t\left(\delta_c \otimes \mathbb{1}_{M_2} + \delta_h \otimes \mathbb{1}_{M_2} + i\left[\frac{1}{2}\varsigma_\rho^{-i\alpha}(V) \otimes P_e + \frac{1}{2}\varsigma_\rho^{i\alpha}(V) \otimes P_g,\ \cdot\right]\right)$,
- $\nu_{a,t} = \nu \otimes \rho_a \circ \tau_a^t|_{M_2}$.

Proposition

$$\nu_{a,t} = \frac{1}{2}\begin{pmatrix} \frac{1}{\mathcal{G}_{\nu,t}(\alpha)} & \mathcal{G}_{\nu,t}(\alpha) \\ & 1 \end{pmatrix}$$

with $\mathcal{G}_{\nu,t}(\alpha) = \nu([D\rho_{-t} : D\rho]^*_{\alpha/2}[D\rho_{-t} : D\rho]_{\alpha/2})$.

Remark
The function $\mathcal{G}_{\nu,t}$ is accessible by tomography of the qbit.

Ancilla tomography definition

- If $\nu = \rho$, $\mathcal{G}_{\rho,t}(\alpha) = \mathcal{F}_{\rho,t}(\alpha)$
- The function $\mathcal{G}_{\nu,t}$ is a Fourier transform,
- But it may not be the characteristic function of a random variable,
- $(\tau_\alpha^t)_t$ differs from $(\tau^t)_t$ only through the local interaction.

Questions:

- Is this method of estimation stable with respect to initial time?
- What can we say when $\nu = \rho_+$?

Ancilla tomography stability

I assume standard hypotheses of the Liouvillean spectral approach to existence and uniqueness of NESS (developed at the end of the 90's and in the 2000's).

Theorem (B., Bruneau, Jakšić, Panati, Pillet 2024)

$$\lim_{t\to\infty}\lim_{T\to\infty}\frac{1}{t}\log\mathcal{G}_{\rho_T,t}(\alpha)=\lim_{T\to\infty}\lim_{t\to\infty}\frac{1}{t}\log\mathcal{G}_{\rho_T,t}(\alpha)=\lim_{t\to\infty}\frac{1}{t}\log\mathcal{F}_{\rho,t}(\alpha).$$

We have stability, however, $\mathcal{G}_{\rho_T,t}$ cannot be interpreted as a Laplace transform beyond this stability.

Two one parameter families of Liouvilleans:

- For any $A \in \mathcal{O}$, $\pi(\tau^t(A)) = e^{it\mathcal{L}_\alpha}\pi(A)e^{-it\mathcal{L}_\alpha}$ with

$$\mathcal{L}_\alpha = \mathcal{L} - J\pi(V)J + J\pi(\varsigma_\rho^{-i\overline{\alpha}}(V))J$$

  Remark that $L_0 = L$,
- $\widehat{\mathcal{L}}_\alpha = \Delta_\rho^{-\alpha/2}\mathcal{L}_{1/2-\alpha}\Delta_\rho^{\alpha/2}$, or

$$\widehat{L}_\alpha = \mathcal{L} - \pi(V) + J\pi(V)J + \pi(\varsigma_\rho^{i\alpha/2}(V)) - J\pi(\varsigma_\rho^{-i(1-\overline{\alpha})/2}(V))J$$

Proposition

$$\mathcal{F}_{\rho,t}(\alpha) = \langle \Omega_\rho, e^{it\mathcal{L}_{1/2-\alpha}}\Omega_\rho\rangle,$$

$$\mathcal{G}_{\rho_T,t}(\alpha) = \langle \Omega_\rho, e^{iT\mathcal{L}_{1/2}}e^{it\widehat{\mathcal{L}}_\alpha}\Omega_\rho\rangle.$$

Heat fluctuations heavy tails

We now measure

$$H_h + H_c$$

twice and take the difference: **Heat variation** Q_t.

Theorem (B., Panati, Raquepas '19)
If $|\langle E|V|E'\rangle|^2 = O(|E' - E|^{2n})$*, then*

$$\sup_t \langle Q_t^{2+2n} \rangle_\rho < \infty.$$

If $|\langle E|V|E'\rangle|^2 = O(e^{\gamma|E'-E|})$*, then*

$$\sup_t \langle e^{\gamma|Q_t|} \rangle_\rho < \infty.$$

Theorem (B., Panati, Raquepas '19)
There exist examples (quasi-free fermions and bosons) such that for almost any time t*,*

$$\langle Q_t^4 \rangle_\rho = \infty.$$

Some open questions

- Taking into account imperfect measurements of $-\log \rho$ and check the consistency of a perfect measurement limit?
- What does it says about energy eigenstates?
- Other definitions of stochastic entropy production? Continuous measurement in a non-Markovian setting?

Thank you!