Deviation bounds and concentration inequalities for quantum noises (arXiv:2109.13152)

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Classical estimation

State space: Ω (finite)

Distribution: Probab. meas. $\pi \in \mathcal{P}(\Omega)$

Goal: estimate π.

I.i.d.: $(Y_k)_{k\in\mathbb{N}}$, $Y_k \sim \pi$

Markov process: $(X_t)_{t\in\mathbb{R}_+}$ with generator L, unique invariant measure $\pi L = 0$.

Two estimators of π:

$$E_n := \frac{1}{n}\sum_{k=1}^{n}\delta_{Y_k}.$$

$$E_t := \frac{1}{t}\int_0^t \delta_{X_s}\,\mathrm{d}s.$$

Deviation bounds

Sanov's theorem:

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(\mathrm{d}(E_n, \mu) \le R) = - \inf_{\nu:\mathrm{d}(\nu,\mu)\le R} D(\nu|\pi).$$

$$D(\nu|\pi) := \sum_{i\in\Omega} \nu(i) \log \frac{\nu(i)}{\pi(i)}.$$

Donsker-Varadan's theorem:

$$\lim_{n\to\infty} \frac{1}{t} \log \mathbb{P}(\mathrm{d}(E_t, \mu) \le R) = - \inf_{\nu:\mathrm{d}(\nu,\mu)\le R} I(\nu).$$

$$I(\nu) := \sup_{a>0} \left(- \sum_{i\in\Omega} \nu(i) \frac{(La)(i)}{a(i)} \right).$$

Definition (Detailed balance condition)
The Markov generator L is said to verify detailed balance if it is self-adjoint with respect to the inner product $\langle a, b\rangle_\pi := \mathbb{E}_\pi(ab) = \sum_{i\in\Omega} \pi(i)a(i)b(i)$.

Equivalently if $\pi(i)L_{ij} = \pi(j)L_{ji}$.

Definition
Dirichlet form

$$\mathcal{E}_L(a) = -\langle a, La\rangle_\pi$$

Theorem (Deuschel, Stroock '01)
If L verifies the detailed balance condition, $I(\nu) = \mathcal{E}_L(a)$ with $a = \sqrt{d\nu/d\pi}$

Moreover, for any function $f : \Omega \to \mathbb{R}$, any initial distribution ν and $r \geq 0$,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_\nu \left(\frac{1}{t} \int_0^t f(X_s) - \mathbb{E}_\pi(f) > r \right) = -I_f(\mathbb{E}_\pi(f) + r)$$

with I_f the lower semi-continuous regularization of

$$J_f : x \mapsto \inf_{\|g\|_{L^2(\pi)}=1} \left\{ \mathcal{E}_L(g) : \mathbb{E}_\pi(fg^2) = x \right\}.$$

Definition (Logarithmic Sobolev inequality and constant)
Let $f = \sqrt{d\nu/d\pi}$. Logarithmic Soboloev inequality: $\alpha D(\nu|\pi) \leq \mathcal{E}_L(f)$

Logarithmic Sobolev constant: $\alpha(L)$ maximal α such that the inequality holds.

Definition (Transportation-information inequality)
For $\nu \ll \pi$ let $f = \sqrt{d\nu/d\pi}$. There exist $C > 0$ such that for any such ν,

$$W_1(\nu, \pi) \leq \sqrt{2C\mathcal{E}_L(f)}$$

Proposition (Guillin, Léonard, Wu, Yao '09)
Let $f = \sqrt{d\nu/d\pi}$. If the transportation-information inequality holds with constant $C > 0$, then for any Lipschitz function g,

$$\mathbb{P}_\nu\left(\frac{1}{t}\int_0^t g(X_s)ds - \mathbb{E}_\pi(g) > r\right) \leq \|f\|_{L^2(\pi)} \exp\left(-\frac{tr^2}{2C\|g\|_{\mathrm{Lip}}^2}\right).$$

Quantum estimator

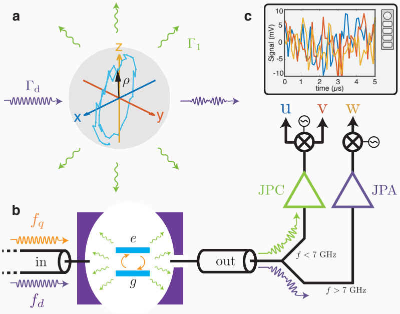
Hilbert space: $\mathcal{H}$ (finite dimension)

Density operators: $\sigma \in \mathcal{D}(\mathcal{H})$

Goal: estimate σ.

Estimators?

B. Huard's group experiment (ENS Lyon)



Quantum Markov dynamics

Dynamics: Semigroup of unital completly positive maps

$$e^{t\mathcal{L}} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$$

$e^{t\mathcal{L}}\mathrm{Id} = \mathrm{Id}.$

Theorem (Gorini, Kossakowski, Sudarshan '76 and Lindblad '76)
There exists $H \in \mathcal{B}(\mathcal{H})$ such that $H = H^*$ and $\mathbf{L} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \otimes \mathbb{C}^k$ such that,

$$\mathcal{L} : A \mapsto i[H, A] + \mathbf{L}^*(A \otimes \mathrm{Id}_{\mathbb{C}^k})\mathbf{L} - \tfrac{1}{2}(\mathbf{L}^*\mathbf{L}A + A\mathbf{L}^*\mathbf{L}).$$

Definition (Primitive semigroup)
The semigroup $(e^{t\mathcal{L}})_t$ is primitive if there exists a unique $\sigma \in \mathcal{D}$ positive definite such that $\mathrm{tr}(\sigma e^{t\mathcal{L}}(A)) = \mathrm{tr}(\sigma A)$ for any $A \in \mathcal{B}(\mathcal{H})$. Or equivalently $\mathrm{tr}(\sigma \mathcal{L}(A)) = 0$ for any $A \in \mathcal{B}(\mathcal{H})$.

From now on we always assume $(e^{t\mathcal{L}})$ is primitive with unique invariant density operator $\sigma > 0$.

There exists a Markov process $(\rho_t)_t$ taking value in $\mathcal{D}$, called quantum trajectory, such that

$$\mathbb{E}_\rho(\mathrm{tr}[\rho_t A]) = \mathrm{tr}[\rho e^{t\mathcal{L}} A]$$

for any $A \in \mathcal{B}(\mathcal{H})$ and there exists $u \in S^{k-1}(\mathbb{C})$ such that the measured signal $(X_t(u))_t$ is given by

$$X_t(u) = \int_0^t \mathrm{tr}(O(u)\rho_s)ds + W_t$$

with

$$O(u) = u.\mathbf{L} + (u.\mathbf{L})^* = \sum_{i=1}^{\dim \mathcal{H}} u_i L_i + \bar{u}_i L_i^*.$$

The derivation of these processes follows from quantum stochastic calculus.

Remark: The process (ρ_t) is Markov but not $(X_t(u))$.

Theorem (Kümmerer, Maassen '04)
Almost surely,

$$\lim_{t\to\infty} \frac{1}{t} X_t(u) = \mathrm{tr}(O(u)\sigma)$$

Deviation bounds

There exists a quantum equivalent of Sanov's theorem for i.i.d estimation of σ with rate

$$S(\rho|\sigma) = \mathrm{tr}[\rho(\log\rho - \log\sigma)].$$

And an equivalent of Donsker-Varadan's theorem for $(X_t(u))$:

Theorem (Jakšić, Pillet, Westrich. '14)
There exists a good rate function I such that full large deviation principle holds:

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_\rho \left(\tfrac{1}{t} X_t(u) > \mathrm{tr}(O(u)\sigma) + r \right) \sim -I(\mathrm{tr}(O(u)\sigma) + r).$$

Can we relate I to a non-commutative version of Dirichlet's form?

Definition (KMS inner product)

$$\langle A, B\rangle_\sigma = \mathrm{tr}(\sigma^{\frac{1}{2}} A^* \sigma^{\frac{1}{2}} B).$$

Definition ((symmetrized) Dirichlet's form)

$$\mathcal{E} : A \mapsto \tfrac{1}{2}(\langle A, \mathcal{L}(A)\rangle_\sigma + \langle \mathcal{L}(A), A\rangle_\sigma).$$

Theorem (B., Hänggli, Rouzé '21)
Let $\Phi_u(A) = Au.\mathbf{L} + (u.\mathbf{L})^* A$ *and* $f_u(A) = \frac{1}{2}(\langle A, \Phi_u(A)\rangle_\sigma + \langle \Phi_u(A), A\rangle_\sigma)$*. Then, for any* $t, r \geq 0$ *and* $\rho \in \mathcal{D}$*, setting* $D = \sigma^{-\frac{1}{4}} \rho^{\frac{1}{2}} \sigma^{-\frac{1}{4}}$*,*

$$\mathbb{P}_\rho(\tfrac{1}{t} X_t(u) > \mathrm{tr}(O(u)\sigma) + r)$$
$$\leq \|D\|_{L^2(\sigma)} \exp\left(-t \inf_{\|A\|_{L^2(\sigma)}=1} \{\mathcal{E}(A) + \tfrac{1}{2}(f_u(A) + \mathrm{tr}(O(u)\sigma) + r)^2\}\right).$$

We have a deviation bound interpretation of non commutative Dirichlet's form. Compared to the classical bound not optimisation on a subset of measures but a quadratic penalization term.

Proof: Based on quantum stochastic calculus, Markov's inequality, Ando-Lieb concavity theorem and a minimax theorem.

Definition (KMS Quantum Detailed Balance (KMS QDB))
The semigroup $e^{t\mathcal{L}}$ *(or equivalently* $\mathcal{L}$*) verifies KMS QDB if* $\mathcal{L}$ *is symmetric with respect to KMS inner product.*

Proposition (Fagnola, Umanità '10, Amorim, Carlen '21)
If $\mathcal{L}$ *verifies KMS QDB, then there exists* $u \in S^{k-1}(\mathbb{C})$ *such that* Φ_u *as defined in previous theorem is symmetric with respect to KMS inner product.*

Theorem (B., Hänggli, Rouzé '21)
Assume $\mathcal{L}$ *verifies KMS QDB and* $u \in S^{k-1}(\mathbb{C})$ *is such that* Φ_u *is KMS symmetric. Then, the rate function* I *for* $(X_t(u))_t$ *LDP is such that*

$$I(x) = \inf_{\|A\|_{L^2(\sigma)}=1} \{\mathcal{E}(A) + \tfrac{1}{2}(f_u(A) + x)^2\}.$$

We have saturation of the deviation bound depending on Dirichlet's form.

To discuss functional inequalities and concentration bounds we require a more restrictive notion of detailed balance.

Definition (GNS QDB)
The generator $\mathcal{L}$ (or the semigroup it generates) verifies GNS QDB if it is symmetric with respect to the inner product

$$(A, B) \mapsto \mathrm{tr}[\sigma A^* B].$$

Theorem (Fagnola, Umanità '10)
Assume $\mathcal{L}$ verifies GNS QDB. Then it verifies KMS QDB.

The reversed implication does not hold (Carlen, Maas '17 and B., Cuneo, Jaksic, Pillet '22).

Example of GNS QDB semigroup: Depolarizing channels, Davies generators.

Remark: We also have GNS QDB $\implies$ BKM QDB and not the opposite (Carlen, Maas '17). However KMS and BKM notions of QDB are not comparable (B., Hänggli, Rouzé '21).

Definition (Logarithmic Sobolev inequality and constant)
For $\rho \in \mathcal{D}$, let $D = \sigma^{-\frac{1}{4}}\rho^{\frac{1}{2}}\sigma^{-\frac{1}{4}}$. Logarithmic Sobolev inequality: $\alpha_2 S(\rho|\sigma) \leq \mathcal{E}(D)$.

The best constant $\alpha_2(\mathcal{L})$ is the logarithmic Sobolev constant.

The right hand side has now a statistical interpretation.

Definition (Transportation cost-information inequality)
There exists $C > 0$ such that for any $\rho \in \mathcal{D}$, setting $D = \sigma^{-\frac{1}{4}}\rho^{\frac{1}{2}}\sigma^{-\frac{1}{4}}$,

$$W_{1,\mathcal{L}}(\rho, \sigma) \leq \sqrt{2C\mathcal{E}(D)}.$$

Remark: The Wassestein distance we consider here depends on a choice of **L** and requires GNS QDB.

Theorem (B., Hänggli, Rouzé '21)
Assume transportation cost-information inequality holds with constant $C > 0$. Then, for any $u \in S^{k-1}(\mathbb{C})$, $\rho \in \mathcal{D}$ and $t, r \geq 0$,

$$\mathbb{P}_\rho\left(\frac{1}{t} X_t(u) > \mathrm{tr}(\sigma O(u)) + r\right) \leq \|D\|_{L^2(\sigma)} \exp\left(-\frac{tr^2}{4C\|\sigma^{\frac{1}{4}}(u.\mathbf{L})^*\sigma^{-\frac{1}{4}} + \sigma^{-\frac{1}{4}} u.\mathbf{L}\sigma^{\frac{1}{4}}\|^2_{\mathrm{Lip}}}\right)$$

with $D = \sigma^{-\frac{1}{4}}\rho^{\frac{1}{2}}\sigma^{-\frac{1}{4}}$.

Remark: Here the Lipschitz norm depends on $\mathbf{L}$ and requires GNS QDB for its definition.

Further remarks

- We can deal with multidimensional diffusions.
- We have similar results including Poisson processes for $(X_t(u))$.
- The upper deviation bounds should still hold for $\mathcal{H}$ infinite dimensional under the appropriate assumptions.
- Our proofs are based on a non commutative extension of stochastic calculus and Girsanov transformation however everything can be done using classical stochastic calculus.

Thank you!