Appearance of particle tracks in detectors

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Particle tracks in a detector

Cloud chamber - Michael F. Schönitzer (CC BY-SA 4.0) - video extract

Position-Velocity Heisenberg's indeterminacy principle

$$\sigma_x(\Psi)\sigma_v(\Psi) \geq \frac{\hbar}{2m}.$$

- No physical state with definite position or velocity.
- No measurement instrument resulting in definite position or velocity of a quantum particle.

How do particle tracks appear?

Question dates back at least to (Mott, 1929).

In the remainder I fix $\hbar = 1$.

Approximate position measurement

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2. Posterior state:
$$\Psi_0'(Q_0) := \frac{F(\hat{x},Q_0)\Psi_0}{\|F(\hat{x},Q_0)\Psi_0\|}.$$

A repeated measurement detector model

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Result: Random sequence of approximate positions: $(Q_n)_{n\in\mathbb{N}_0}$ with law

$$\frac{\mathrm{d}\mathbb{P}_{\Psi_0}(q_0,\ldots,q_n)}{\mathrm{d}q_0\cdots\mathrm{d}q_n} = \|UF(\hat{x},q_n)\cdot UF(\hat{x},q_0)\Psi_0\|^2.$$

Linear dynamics and gaussian approximate measurement

Assume **linear dynamics**: $\begin{pmatrix} U^* \hat{x} U \\ U^* \hat{v} U \end{pmatrix} = S \begin{pmatrix} \hat{x} \\ \hat{v} \end{pmatrix}$ with S a symplectic matrix with bloc decomposition on $\mathbb{R}^3_x \oplus \mathbb{R}^3_v$,

$$S = \begin{pmatrix} S_{xx} & S_{xv} \\ S_{vx} & S_{vv} \end{pmatrix}.$$

Examples: Free particle, Harmonic oscillator, Charged particle in a constant magnetic field.

Assume **Gaussian approximate position measurement**: $\mathcal{L}(Q|x) = \mathcal{N}(x, \Sigma)$,

$$F(x, q) = ((2\pi)^d \det \Sigma)^{-1/4} \exp(-\frac{1}{4}(x - q).\Sigma^{-1}(x - q)).$$

Gaussian track

Theorem (Ballesteros, B., Fraas, Fröhlich 2021)
Assume S_{xv} is invertible. Then, there exists K a 6×3 matrix and $\widehat{\Sigma} > 0$ a 3×3 matrix such that the process

$$\begin{pmatrix} x_{n+1} \\ v_{n+1} \end{pmatrix} = S\left(\begin{pmatrix} x_n \\ v_n \end{pmatrix} + K\eta_n\right)$$

with $(\eta_n)_{n\in\mathbb{N}_0}$ an i.i.d. sequence where $\eta_n \sim \mathcal{N}(0,\widehat{\Sigma})$ and $(x_0, v_0) \sim |\langle\phi_{x,v}|\Psi_0\rangle|^2 \mathrm{d}x\mathrm{d}v$ where $\{\phi_{x,v}\}$ is a set of coherent states, verifies

$$(Q_n)_n \sim (x_n - \eta_n)_n$$

Corollary
Under the same assumptions, if $\mathbb{E}(\|x_0\| + \|v_0\|) < \infty$,

$$\mathbb{E}(Q_n) = \begin{pmatrix} \mathrm{Id} & 0 \end{pmatrix} S \begin{pmatrix} \langle\Psi_0|\hat{x}\Psi_0\rangle \\ \langle\Psi_0|\hat{v}\Psi_0\rangle \end{pmatrix}.$$

Remarks: No restriction on the initial particle state and no extra approximations.

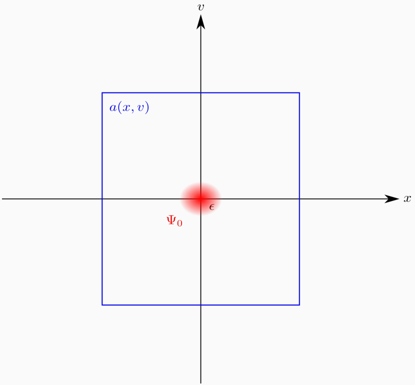
The particle position depends on the past measurement results.

Massive and high energy particles setting

Appropriate scaling: $\sigma_x(\Psi_0)\sigma_v(\Psi_0) \sim \epsilon$ with $\epsilon \ll 1$ (expl. $\epsilon = 1/m$, $m \gg 1$).

$$\text{Typical state scaling: } \Psi_0(x) = \epsilon^{-3/4}\psi\left(\frac{x-x_0}{\sqrt{\epsilon}}\right)e^{iv_0.x/\epsilon}.$$

Observables symbols: $\mathcal{S} := \{a \in C^\infty(\mathbb{R}^3_x \oplus \mathbb{R}^3_v) : \|\partial^\alpha a\|_\infty < \infty, \forall\alpha\}$.



Quantization: $\hat{a} = \mathrm{Op}_\epsilon(a)$ such that $\lim_{\epsilon\downarrow 0} \|\mathrm{Op}_\epsilon(a)\mathrm{Op}_\epsilon(b) - \mathrm{Op}_\epsilon(ab)\| = 0$.

Massive and high energy particles assumptions

Approximate position measurement instrument assumptions:

1. **(Right scale)** For a.e. $q \in \mathbb{R}^3$, $F_q : (x, v) \mapsto F(x, q) \in \mathcal{S}$,
2. **(Stochasticity)** For any ϵ small enough, $\int_{\mathbb{R}^3} \hat{F}_q^* \hat{F}_q \mathrm{d}q = \mathrm{Id}_{L^2(\mathbb{R}^3)}$,
3. **(Technical)** The function $q \mapsto \|F_q\|_\infty^2$ is locally integrable.

Dynamical assumption: Becomes classical (Egorov's theorem). There exists an Hamiltonian diffeomorphism ϕ such that for any $a \in \mathcal{S}$, $a \circ \phi \in \mathcal{S}$ and

$$\lim_{\epsilon \downarrow 0} \| U^* \mathrm{Op}_\epsilon(a) U - \mathrm{Op}_\epsilon(a \circ \phi) \| = 0.$$

Remark: Dynamical assumption verified without limits for linear dynamics (massive free particle, massive harmonic oscillator, massive charged particle in a strong magnetic field: '$\beta = B/m$ fixed').

Track in the high energy classical limit

Theorem (B., Fraas, Fröhlich 2021)
With the assumptions above, if there exists a probability measure μ_0 over $\mathbb{R}^3_x \oplus \mathbb{R}^3_v$ such that for any $a \in \mathcal{S}$,

$$\lim_{\epsilon \downarrow 0} \langle \Psi_0 | \hat{a} \Psi_0 \rangle = \int_{\mathbb{R}^3_x \oplus \mathbb{R}^3_v} a(x, v) \mathrm{d}\mu_0(x, v),$$

Then,

$$(Q_n)_n \xrightarrow[\epsilon \downarrow 0]{\mathcal{L}} (Q(\phi^n(x_0, v_0)))_n$$

with $(x_0, v_0) \sim \mu_0$ and each copy of $Q(x, v)$ independent with law

$$Q(x, v) \sim |F(x, q)|^2 \mathrm{d}q.$$

Corollary (Translation invariant instrument)
Under the same assumptions, if there exists G such that $F(x, q) = G(x - q)$,

$$(Q_n)_n \xrightarrow[\epsilon \downarrow 0]{\mathcal{L}} (x_n + \eta_n)_n$$

with $(x_n, v_n) = \phi^n(x_0, v_0)$ and $(\eta_n)_n$ an i.i.d. sequence of random variable with $\eta_n \sim |G(q)|^2 \mathrm{d}q$.

On going and future work

- Estimation of the initial data,
- Open systems,
- Joint limit with precise instrument,
- Continuous approximate measurement.

Thank you!