Introduction to quantum mechanics for probabilists

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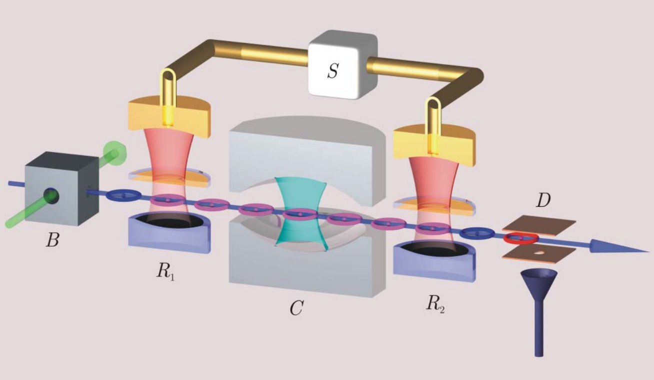
A typical experiment (Serge Haroche's group)



Data:

j	11011111111110011101101111
i	*ddcbccabcdaadaabadddbadbc*
j	0101001101010101101011111
i	*dababbaacbccdadccdcbaaacc*

j	0001000110110000001010110
i	*ddcaddabbccdccbcdaabbccab*
j	0001010100000100011101101
i	*bcdaddaabbbbdbdcdccadaada*

Images: LKB ENS

Modelization

	Classic	Quantic (non commutative)
State space	$\{1, \ldots, d\}$	Hilbert space $\mathcal{H} = \mathbb{C}^d$
"Distributions'	Probability measures: $\mathcal{P}_d$	Density matrices
Evolution	Stochastic matrices	Quantum channels

Definition (Density matrices)

$$\mathcal{S} = \{\rho \in M_d(\mathbb{C}) : \rho \geq 0, \operatorname{tr} \rho = 1\}$$

Definition (Quantum channels)

$\Phi : M_d \to M_d$ *tel que* $\operatorname{tr} \circ \Phi(X) = \operatorname{tr}(X)$ *et* $\exists k \in \mathbb{N}$ *et* $(V_i)_{i=1}^k \in M_d^k$ *such that*

$$\sum_{i=1}^k V_i^* V_i = I_d \quad et \quad \Phi : X \mapsto \sum_{i=1}^k V_i X V_i^* .$$

Remark

$$\Phi(\mathcal{S}) \subset \mathcal{S}$$

Classical inclusion

Let $\iota : \mathcal{P}_d \to \mathcal{S}$ be defined by

$$\iota(p_1, \dots, p_d) = \begin{pmatrix} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & \cdots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \cdots & p_d \end{pmatrix}$$

Proposition

$\iota(\mathcal{P}_d) \subsetneq \mathcal{S}$ and for any stochastic matrix P, there exists a quantum channel Φ_P such that for any $\mathbf{p} \in \mathcal{P}_d$,

$$\iota(\mathbf{p}P) = \Phi_P(\iota(\mathbf{p})).$$

Generalization of Markov chains

Theorem (Perron-Frobenius)

$\exists ! \pi \in \mathcal{P}_d$ such that $\pi > 0$ and $\pi P = \pi$ iff.

$$(\mathrm{id}_{M_d} + P)^{d-1} x > 0, \quad \forall x \geq 0, x \neq 0$$

or equivalently, $\forall i, j \in \{1, \dots, d\}$, there exists $(i_1, \dots, i_p) \in \{1, \dots, k\}^p$, $p \in \mathbb{N}$ such that

$$p_{ii_1} p_{i_1 i_2} \cdots p_{i_p j} > 0.$$

Theorem (Perron-Frobenius (Evans, Høegh-Krohn 1978))

$\exists ! \rho \in \mathcal{S}$ such that $\rho > 0$ and $\Phi(\rho) = \rho$ iff.

$$(\mathrm{id}_{M_d} + \Phi)^{d-1}(X) > 0, \quad \forall X \geq 0, X \neq 0$$

or equivalently, $\forall \rho_1, \rho_2 \in \mathcal{S}$, there exists $(i_1, \dots, i_p) \in \{1, \dots, k\}^p$, $p \in \mathbb{N}$ such that

$$\mathrm{tr}(\rho_2 V_{i_p} \cdots V_{i_1} \rho_1 V_{i_1}^* \cdots V_{i_p}^*) > 0.$$

Measurement result sequence law

Definition

For $\rho \in \mathcal{S}$, $\mathbb{P}_\rho$ is the probability measure over $\Omega = \{1, \dots, k\}^{\mathbb{N}}$ such that,

$$\mathbb{P}_\rho(\omega_1 = i_1, \dots, \omega_n = i_n) = \mathrm{tr}(V_{i_n} \cdots V_{i_1} \rho V_{i_1}^* \cdots V_{i_n}^*).$$

Example

- $\mathbb{P}_\rho = q_0(1)\mathbb{Q}_1 + \cdots + q_0(d)\mathbb{Q}_d$ *with, for all* $\alpha \in \{1, \dots, d\}$, $\mathbb{Q}_\alpha$ *the law of a sequence of iid. random variables valued in* $\{1, \dots, k\}$,
- $\mathbb{P}_\rho$ *can be the law of a Markov chain on* $\{1, \dots, k\}$,
- $\mathbb{P}_\rho$ *can be the law of a Hidden Markov chain,*
- $\mathbb{P}_\rho$ *is a Kusuoka measure used in the study of fractal sets.*

Markov chains on density matrices

Definition

The process $(\rho_n)_{n\in\mathbb{N}}$ *defined by* $\rho_0 = \rho$ *and*

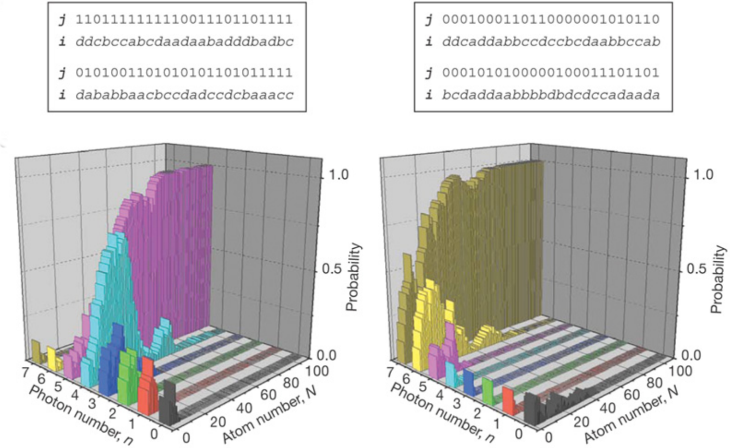
$$\rho_n = \frac{V_{\omega_n}\rho_{n-1}V_{\omega_n}^*}{\mathrm{tr}(V_{\omega_n}\rho_{n-1}V_{\omega_n}^*)}$$

is, with respect to $\mathbb{P}_\rho$*, a Markov chain valued in* $\mathcal{S}$*. Its kernel is*

$$\Pi f(\rho) = \sum_{i=1}^{k} f\left(\frac{V_i\rho V_i^*}{\mathrm{tr}(V_i\rho V_i^*)}\right)\mathrm{tr}(V_i\rho V_i^*).$$

One research subject: The study of the measures $\mathbb{P}_\rho$ and their related Markov chain.

Back to the experiment



Displayed quantity: $((\rho_n)_{ii})_{i=1}^{d}$ as a function of time (or number of measures).

Images: LKB ENS

Non demolition measurements

Theorem (Bauer, Bernard 2011; Bauer, B., Bernard 2013)

If for any $i \in \{1, \dots, k\}$, V_i is diagonal, then there exists $(\mathbb{Q}_i)_{i=1}^d$ some lowas of iid. random variables valued in $\{1, \dots, k\}$ such that

$$\mathbb{P}_\rho = \sum_{i=1}^{d} q_0(i)\mathbb{Q}_i$$

where $q_0(i) = \rho_{ii}$.
Moreover, if for any $i \neq j$, $\mathbb{Q}_i \neq \mathbb{Q}_j$,

$$\rho_n \xrightarrow[n\to\infty]{a.s.} E_{II}$$

with $\text{Proba}(I = i) = \rho_{ii}$.

Proof idea.

For any $i \in \{1, \dots, d\}$, $((\rho_n)_{ii})_{n\in\mathbb{N}}$ is a bounded martingale. Doob's martingale convergence theorem gives the result. □

Remark

This shows that these indirect measurements reproduce the wave function collapse postulate.

Ergodicity

Let $\rho \in \mathcal{S}$ be such that $\rho > 0$ and $\Phi(\rho) = \rho$. Then $\mathbb{P}_\rho$ is invariant with respect to the left shift $\phi(\omega)_n = \omega_{n+1}$. The triplet $(\Omega, \phi, \mathbb{P}_\rho)$ defines a dynamical system.

Theorem

If ρ is the unique element of $\mathcal{S}$ such that $\Phi(\rho) = \rho$, then, $(\Omega, \phi, \mathbb{P}_\rho)$ is ergodic.

Finer study:

- LLN, CLT et LIL;
- Large deviations;
- Hypothesis testing and parameter estimation;
- Relationship with other dynamical system (Gibbs measures);
- Multi-fractal spectrum. . .

Uniqueness of the invariant measure for the Markov chain

Theorem (Purification – Kümmerer, Maassen 2005)

Under some relatively tractable assumptions on $\{V_i\}_{i=1}^k$,

$$\operatorname{spec}\rho_n \xrightarrow[n\to\infty]{a.s.} \{0,1\}.$$

Theorem (B., Fraas, Pautrat, Pellegrini 2017)

If Φ *has a unique fixed point in* $\mathcal{S}$ *and* $\{V_i\}_{i=1}^k$ *verifies purification, then there exists a unique probability measure* ν *over* $\mathcal{S}$ *such that*

$$\mathbb{E}(f(\rho_1)|\rho_0 \sim \nu) = \mathbb{E}(f(\rho_0)|\rho_0 \sim \nu)$$

for any continuous function $f : \mathcal{S} \to \mathbb{R}$.

Study of the Markov chain

As for the uniqueness of the invariant measure, it requires an original approach.

Construction of estimators of ρ_n as functions of $\omega \in \Omega$ independent of ρ_0.

Finer study:

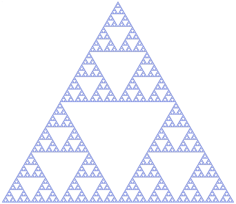
- LLN, CLT, LIL;
- Large deviations;
- Reversibility and quantum detailed balance;
- Properties of the invariant measure (can be quite irregular);
- Generic behavior;
- Feedback control (preparation of non classical state like Schrödinger's cat)...

Examples of other motvations

- Hidden Markov Models in statistics;
- The measures $\mathbb{P}_\rho$ are Gibbs measures with eventually non continuous potential. However, the thermodynamic formalism can still be used to study them.
- Definition of relevant measures on fractals (Kusuoka 1989).
  Example on Sierpiǹski's triangle $\mathcal{X}$. Let $k = 3$ and $\pi : \Omega \to \mathcal{X}$ be defined by

  $$\pi(\omega) = \sum_{n=0^\infty} 2^{-n} e_{\omega_n}, \quad e_1 = (0,0), e_2 = (1,0) \text{ et } e_3 = (1, \sqrt{3})/2.$$

  Then $\mathbb{P}_\rho \circ \pi^{-1}$ defines a measure on $\mathcal{X}$. Kusuoka's goal was to define over $\mathcal{X}$ the equivalent of a Laplacian.



Credits: Wikimedia user "Marco Polo".

Probability used in quantum mechanics (I. Nechita)

Probabilities are used to:

- Study random quantum channels and their generic properies (spectral gap, invariant state . . .
- Show the existence of counter examples to conjectures

Non additive capacity

Definition (Classical capacity)

The classicla capacity of Φ is a non negative umber $\chi(\Phi)$ that quantifies the quantity of (classical) information that a quantum channel can transmit.

Proposition (Classical additivity)

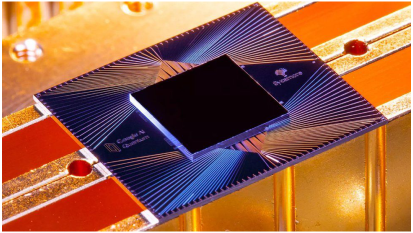
For two classical channels P_1, P_2, $\chi(P_1 \times P_2) = \chi(P_1) + \chi(P_2)$.

Theorem (Quantum non additivity (Hastings 2008; Collins, Nechita 2009))

For D large enough, there exists $\Phi_1 : M_D \to M_D$ and $\Phi_2 : M_D \to M_D$ two quantum channels such that $\chi(\Phi_1 \otimes \Phi_2) > \chi(\Phi_1) + \chi(\Phi_2)$.

Remark: The proofs use techniques in free probability and random matrix theory.

Quantum computing and quantum supremacy



Credits: Google

Google experiment: Sampling a probability measure hard to sample with a classical computer.
Big Hilbert space: $d = 2^{53}$ ($\sim 10^{16}$ param.)

$$\rho_n = U_n \cdots U_1 \rho U_1^* \cdots U_n^*$$

with $U_1, \dots, U_n$ random unitary matrices.
In the experiment: sampling of $((\rho_n)_{ii})_{i=1}^{2^{53}}$.
Mathematics: Properties of the random state ρ_n? **Random quantum circuit** (I. Nechita).

Operator algebra proint of view

Back to the non commutative generalization of classical probabilities.

Definition

A unital C^*-algebra $\mathcal{A}$ is an algebra containing an element 1, equipped with an anti-linear involution $*$ and a norm $\|\cdot\|$, such that $\|a^*a\| = \|a\|^2$ for any $a \in \mathcal{A}$ and such that $\mathcal{A}$ is norm closed.

Definition

A state over a unital C^*-algebra is a positive 1-form ν such that $\nu(1) = 1$.

Example

- *Let F be a compact subset of $\mathbb{R}^n$. Then the set $C^0(F)$ equipped with pointwise product $ab(x) = a(x)b(x)$, the involution $a^*(x) = \overline{a(x)}$ and the sup norm, is a C^*-algebra.*
  The states over $C^0(F)$ are the expectations of random variables taking values in F: $\nu(f) = \mathbb{E}(f(X))$.
- *The algebra of complex $d \times d$ matrices M_d equipped with the conjugation-transposition involution $*$ and the operator 2-norm is a unital C^*-algebra.*
  The states over M_d are defined by density matrices. For any state ν over M_d there exists $\rho_\nu \in \mathcal{S}$ such that $\nu(X) = \mathrm{tr}(\rho_\nu X)$.

Non commutative probability

	Commutative	Non commutative
Observables	Continuous bounded functions	unital C^*-algebra
States	Expectation	Normalized positive 1-form
Evolution	Markov kernel	Quantum channels

These non commutative generalizations of probabilities are studied for their own sake.
In particular, new notions of independence can be defined.
Beyond quantum mechanics, they are used to study random matrices, graphs (traffics) or to construct algebra invariants (free probability) (G. Cebron).

Theorem (Gelfand representation)

A commutative C^*-algebra is *-isomorphic to the space of continuous functions vanishing at infinity over the space of characters equiped with the weak* topology.

Specific questions in quantum mechanics: compatibility

A measurement in quantum mechanics is described by a POVM: $(\Omega, \mathcal{F})$ is a measurable space, $M : \mathcal{F} \to \mathcal{A}_+$ σ-additive such that $M(\Omega) = 1_{\mathcal{A}}$. The triplet $(\Omega, \mathcal{F}, M)$ is a POVM. (Probability measure valued in the positive operators)

For any state ν over $\mathcal{A}$, it defines a probability measure over $(\Omega, \mathcal{F})$:

$$\mathbb{P}_{\nu,M}(A) = \nu(M(A)).$$

Question

Given two POVM $(\Omega_1, \mathcal{F}_1, M_1)$ *and* $(\Omega_2, \mathcal{F}_2, M_2)$ *defined on* $\mathcal{A}$*, does there exists a POVM* $(\Omega 1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, M_{12})$ *such that*

$$\mathbb{P}_{\nu,M_{12}}(A \times \Omega_2) = \mathbb{P}_{\nu,M_1}(A)$$

and

$$\mathbb{P}_{\nu,M_{12}}(\Omega_1 \times A) = \mathbb{P}_{\nu,M_2}(A)$$

for any state ν *and any measurable set* A *? (I. Nechita, F. Loulidi)*

Entanglement

Two quantum systems A and B are jointly described by a tensor product of their algebras:

$$\mathcal{A}_{AB} = \mathcal{A}_A \otimes \mathcal{A}_B.$$

The set of states over $\mathcal{A}_{AB}$ is larger than the states defined by

$$\nu_{AB} = \int_\Lambda \nu_{A,\lambda} \otimes \nu_{B,\lambda} d\mu(\lambda). \tag{1}$$

The states that can be written that way (1) are called **separable** the others are said **entangled**.
Let $\mathcal{S}_{sep}$ be the set of states that can be written as in (1) and $\mathcal{S}_{ent} = \mathcal{S} \setminus \mathcal{S}_{sep}$.

Research subjects (I. Nechita):

- Characterization fo $\mathcal{S}_{intr}$ using "entanglement wintess" easily computable or accessible;
- Properties of subsets of $\mathcal{S}_{ent}\ldots$

Quantum weirdness: Teleportation

Transmission of a full quantum state using only 2-bits of classical information and an entangled state.
Alice: Algebra $M_2 \otimes M_2$, a state $x \in M_2$ unkown to Bob,
Bob: Algebra $M_2 \otimes M_2$, a state $x_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ known to Alice,
Alice and Bob: Share an entagled state over $M_2 \otimes M_2 \equiv M_4$: $\psi = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$,
Full state of Alice and Bob: $x \otimes \psi \otimes x_0 \in M_2 \otimes M_2 \otimes M_2 \otimes M_2$.

Protocol:

- Alice measures the POVM $r \mapsto \pi_r \otimes I_2 \otimes I_2$ with $\Omega = \{1, 2, 3, 4\}$ and $\pi_r \in M_2 \otimes M_2$ rank 1 orthogonal projections:

$$x \otimes \psi \otimes x_0 \to \psi_A(r) \otimes \psi_B(r)$$

  avec $\psi_A(r), \psi_B(r) \in M_2 \otimes M_2$;
- Alice communicates classicaly r to Bob;
- Bob updates its state with a r-dependent 4×4 unitary matrix: $\psi_B(r) \to U_r \psi_B(r) U_r^* = \phi_r \otimes x$.
- Bob recovers x.

XOR games (CHSH inequality)

A referee (R) sends two questions (amongst $\{0, 1\}$ with renewal), one to Alice and one to Bob. They answer either by yes (1) or no (-1). Alice and Bob win if they answer the same thing.
A quantity $B(\rho)$ quantifies the correlation and maximal gain Alice and Bob can expect ifthey share a state ρ over their joint algebra $\mathcal{A}_{AB} = \mathcal{A}_A \otimes \mathcal{A}_B$, with $\mathcal{A}_A$ and $\mathcal{A}_B$ two copies of M_2.

Theorem

$$2 = \sup_{\rho \in \mathcal{S}_{sep}} B(\rho) < \sup_{\rho \in \mathcal{S}} B(\rho) = 2\sqrt{2}.$$

Remark

- *Alain Aspect's experiments (1982) showed that $B(\rho) > 2$ and no experiment has ever found $B(\rho) > 2\sqrt{2}$.*
  Concerning the physics it means that local hidden variable theories cannot describe quantum phenomena.
- *It can be generalized to a higher number of questions and answers and to larger set of states $\mathcal{S}_{GPT} \supsetneq \mathcal{S}$. Both questions are then related to convex geometry.*

Researchers in probability, quantum mechanics and quantum information at IMT (and LPT)

Permanent researchers:

- Tristan Benoist (IMT)
- Ion Nechita (LPT)
- Clément Pellegrini (IMT)

PHD. studends:

- Faedi Loulidi (LPT)
- Denis Rochette (IMT)