Quantum trajectories and non i.i.d. random products of matrices

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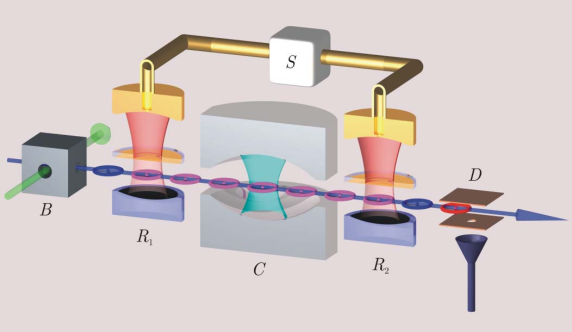
A canonical experiment

S. Haroche group experiment:



j 11011111111110011101101111
i ddcbccabcdaadaabadddbadbc

j 0101001101010101101011111
i dababbaacbccdadccdcbaaacc

j 0001000110110000001010110
i ddcaddabbccdccbcdaabbccab

j 0001010100000100011101101
i bcdaddaabbbbdbdcdccadaada

Pictures: LKB ENS

Average evolution

On finite dimensional Hilbert spaces.

- **Quantum states:** Finite dimensional quantum system state: Density matrices $\mathcal{D}$:

$$\mathcal{D} = \{\rho \in M_d(\mathbb{C}) \,|\, \rho \geq 0, \quad \operatorname{tr} \rho = 1\}.$$

- **Observables:** Self-adjoint matrices: $M_d^{sa}(\mathbb{C})$. To each observable A corresponds a random variable X_A on $\Omega = \{1, \dots, d\}$ whose moments are

$$\mathbb{E}_\rho(X_A^n) = \operatorname{tr}(A^n \rho), \quad \forall n \in \mathbb{N}.$$

- **(Average) Evolution:** A completely positive trace preserving (CPTP) map $\Phi : \mathcal{D} \to \mathcal{D}$.

$$\exists \{V_i\}_{i=1,\dots,\ell} \subset M_d(\mathbb{C}), \quad \sum_{i=1\dots,\ell} V_i^* V_i = I_d$$

such that,

$$\Phi(\rho) = \sum_{i=1}^{\ell} V_i \rho V_i^*.$$

$$\mathbb{E}(\rho_n) = \Phi^n(\rho).$$

Quantum trajectories

Definition (Unraveling)

Given a CPTP map Φ, fix one of its Kraus family $\{V_i\}_{i=1,\dots,\ell}$, then the stochastic process defined by

$$\rho_n = \frac{V_{i_n}\cdots V_{i_1}\rho V_{i_1}^*\cdots V_{i_n}^*}{\mathrm{tr}(V_{i_n}\cdots V_{i_1}\rho V_{i_1}^*\cdots V_{i_n}^*)}, \quad \text{with proba. } \mathbb{P}_\rho(i_1,\dots,i_n) := \mathrm{tr}(V_{i_n}\cdots V_{i_1}\rho V_{i_1}^*\cdots V_{i_n}^*)$$

is called an unraveling of Φ.

Definition (Quantum trajectory)

Let $\{V_i\}_{i=1,\dots,\ell} \subset M_d(\mathbb{C})$ define a CPTP map Φ. Then the Markov process defined by the Kernel,

$$(\Pi f)(\rho) = \sum_{i=1}^{\ell} f\left(\frac{V_i\rho V_i^*}{\mathrm{tr}(V_i\rho V_i^*)}\right)\mathrm{tr}(V_i\rho V_i^*)$$

is called a Quantum trajectory.

Proposition

Quantum trajectories and Unravelings define the same processes and $\mathbb{E}_\rho(\rho_n) = \Phi^n(\rho)$.

The dynamical system picture

ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	$\cdots$	ω_n	$\cdots$

$\omega_n = 1, \ldots, \ell$.

- States of the dynamical system: $\mathcal{A} := \{1, \ldots, \ell\}$
- "Trajectory" space: $\Omega = \{1, \ldots, \ell\}^{\mathbb{N}} \equiv [0, 1]$, time: $n \in \mathbb{N}^*$.
- Probability measure on the "trajectory" space Ω:

$$\mathbb{P}_\rho(\{\omega | \omega_k = i_k, 1 \leq k \leq n\}) = \mathrm{tr}(V_{i_n} \cdots V_{i_1} \rho V_{i_1}^* \cdots V_{i_n}^*),$$

- Time shift: $f \circ \phi^n(\omega_1, \omega_2, \ldots) = f(\omega_{n+1}, \omega_{n+2}, \ldots)$.

$$\mathbb{P}_\rho \circ \phi^{-n} = \mathbb{P}_{\Phi^n(\rho)}.$$

Perron–Frobenius Theorem for CP maps

Theorem (Perron–Frobenius [Evans, Høegh-Krohn '77])

Let $\Phi : M_d(\mathbb{C}) \to M_d(\mathbb{C})$ be a CPTP map. Then the two following statements are equivalent.

(i) **(Irr.)** *If an orthogonal projector P is such that $V_i P\mathbb{C}^d \subset P\mathbb{C}^d$ for all $i \in \{1, \dots, \ell\}$, then $P \in \{0, I_d\}$.*

(ii) *1 is simple eigenvalue of Φ and the corresponding eigenstate is positive definite: $\exists! \rho_{inv.} \in \mathcal{D}$, s.t. $\Phi(\rho_{inv.}) = \rho_{inv.} > 0$.*

Moreover, (i) or (ii) imply the peripheral spectrum of Φ is a finite subgroup of $U(1)$:

$$\mathrm{spec}(\Phi) \cap U(1) = \{e^{i2\pi \frac{k}{m}}\}_{k=1,\dots,m}.$$

There also exist a unitary $U \in M_d(\mathbb{C})$ with spectral decomposition

$$U := \sum_{k=1}^{m} e^{i2\pi \frac{k}{m}} P_k$$

such that $\Phi(P_k) = P_{k-1}$.

Ergodic properties

Proposition (Convergence in total variation)

Assume Φ is irreducible. Then there exists $m \in \{1, \dots, d^2\}$, $C > 0$ and $\lambda < 1$ such that

$$\sup_{\rho \in \mathcal{D}} \sup_{A \subset \Omega} \left| \frac{1}{m} \sum_{k=1}^{m} \mathbb{P}_\rho \circ \phi^{-k-mn}(A) - \mathbb{P}_{\rho_{inv.}}(A) \right| \leq C\lambda^n.$$

Corollary

Assume Φ is irreducible and $m = 1$. Then the dynamical system $(\Omega, \phi, \mathbb{P}_{\rho_{inv.}})$ is exponentially mixing. For any measurable $A, B \subset \Omega$,

$$\left| \mathbb{P}_{\rho_{inv.}}(A\phi^{-n}(B)) - \mathbb{P}_{\rho_{inv.}}(A)\mathbb{P}_{\rho_{inv.}}(B) \right| \leq C\lambda^n.$$

Remark ([Guta, van Horssen '14; Carbone, Pautrat '15])

Using particularly Perron–Frobenius Theorem, the Law of Large Numbers, the Central Limit Theorem and a Large Deviation Principle follows directly for any random variable depending only on finite sequences of $\{1, \dots, \ell\}$ elements.
A Large Deviation Principle also holds for the empirical measure over Ω.

Markov Chain first asymptotic properties

Theorem (First Law of Large Numbers [Kümmerer, Maassen '04])

Let $(\rho_n)_n$ be an unraveling of Φ. Then

$$\rho_\infty := \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \rho_k$$

exists $\mathbb{P}_\rho$–almost surely and is such that $\Phi(\rho_\infty) = \rho_\infty$.
If moreover Φ is irreducible, $\rho_\infty = \rho_{inv}$. $\mathbb{P}_\rho$–almost surely.

Theorem (Purification [Kümmerer, Maassen '04])

Let $\{V_i\}_{i=1,\dots,\ell}$ be a finite family of $d \times d$ complex matrices corresponding to the Kraus decomposition of a CPTP map Φ.
(Pur.) *Assume that any orthogonal projector Q such that $QV_i^* V_i Q \propto Q$ for all $i \in \{1, \dots, \ell\}$ is of rank 1.*
Then the sequence $(\rho_n)_n$ purifies almost surely as $n \to \infty$. Namely, there almost surely exists a sequence of rank 1 orthogonal projectors $(|x_n\rangle\langle x_n|)_n \subset \mathcal{D}$ such that

$$\lim_{n\to\infty} \|\rho_n - |x_n\rangle\langle x_n|\| = 0, \quad \mathbb{P}_\rho - a.s..$$

It implies,

$$\lim_{n\to\infty} S(\rho_n) = \lim_{n\to\infty} -\operatorname{tr}(\rho_n \ln \rho_n) = 0, \quad \mathbb{P}_\rho - a.s..$$

Theorem (Bauer, Bernard '11)

Non demolition: *Assume that there exists an o.n.b. of pointers $\mathcal{P}$ such that all the V_i's are diagonal in $\mathcal{P}$.*
Distinguishability: *Assume that for any two different $x, y \in \mathcal{P}$, there exists i such that $\|V_i x\|_2 \neq \|V_i y\|_2$.*

Then,

$$\lim_{n\to\infty} \rho_n = |\hat{x}\rangle\langle\hat{x}|, a.s.$$

with $\hat{x} : \Omega \to \mathcal{P}$ and

$$\mathbb{P}_\rho(\hat{x} = y) = \mathrm{tr}(|y\rangle\langle y|\rho).$$

Ergodic properties of the Markov Chain

Theorem (B., Fraas, Pautrat, Pellegrini '16)

Assume **(Irr.)** *and* **(Pur.)** *hold. Then,* Π *accepts a unique invariant measure* $\nu_{inv.}$ *and there exists* $m \in \{1, \dots, d^2\}$, $C > 0$ *and* $\lambda < 1$ *such that, for any measure* ν *over* $\mathcal{D}$,

$$W_1\left(\frac{1}{m}\sum_{k=1}^{m} \nu\Pi^{k+mn}, \nu_{inv.}\right) \leq C\lambda^n$$

where W_1 *is the order* 1 *Wasserstein metric.*

Remark

- *No convergence in total variation possible since if* ν_a *has a continuous support,* ν_b *is pure point and all the* V_i*'s are invertible,*

$$\|\nu_a\Pi^n - \nu_b\Pi^n\|_{TV} = 1, \quad \forall n \in \mathbb{N}.$$

- ϕ*–irreducibility methods are not suitable for this Markov Chain.*
- *Proof inspired by product of i.i.d. random matrices.*
- *Under stronger conditions (invertibility, strong irreducibility) Guivarc'h and Le Page proved a similar result in 2004.*
- *Assumptions* **(Irr.)** *and* **(Pur.)** *are optimal. A slightly more general assumption* **(Irr.)** *is necessary.*

Proof structure

Proof.

1. Define $\hat{\mu}_n$ as a sequence of estimates of the initial state given the growing sequence $(\omega_1, \ldots, \omega_n)$:

$$\hat{\mu}_n = \operatorname{argmax}_{\mu \in \mathcal{D}} \ln \mathbb{P}_\mu(\omega_1, \ldots, \omega_n).$$

   Let

$$\hat{\rho}_n := \frac{V_{i_n} \cdots V_{i_1} \hat{\mu}_n V_{i_1}^* \cdots V_{i_n}^*}{\mathrm{tr}(V_{i_n} \cdots V_{i_1} \hat{\mu}_n V_{i_1}^* \cdots V_{i_n}^*)}.$$

   Then **(Pur.)** implies

$$\mathbb{E}_\nu \left(\mathrm{d}(\hat{\rho}_n, \rho_n)\right) \leq C\lambda^n.$$

2. The estimated sequence $(\hat{\rho}_n)$ not depending on the initial state ρ, the convergence follows from the ergodic properties of $\mathbb{P}_{\mathbb{E}_\nu(\rho)}$ implied by **(Irr.)**, the uniformity of the sub geometric bounds in the initial state ρ and the Markov property.
3. The Wasserstein metric bound is obtained by Kantorovich and Rubinstein Duality Theorem.

$\square$

First consequences

Theorem (Law of Large Numbers [BFPP '16])

Assume **(Irr.)** *and* **(Pur.)** *hold. Then for any continuous function on $\mathcal{D}$, for any initial probability measure ν over $\mathcal{D}$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(\rho_k) = \mathbb{E}_{\nu_{inv.}}(f(\rho)), \quad a.s.$$

Theorem (Functional Central Limit Theorem [BFPP '16])

Assume **(Irr.)** *and* **(Pur.)** *hold. Then for any Hölder continuous function g the Functional Central Limit Theorem Holds.*

Current developments

- **Hypothesis testing** Ability to distinguish between different CPTP maps (mutual singularity and error exponents).
  Applied to Hypothesis Testing of the arrow of time [B., Jaksic, Pautrat, Pillet '16].
- **Parameter estimation** [Guta, Kiukas, Levitt '15–'16].
- **Dynamical Phases** Characterisation of the existence of dynamical phases[Guta, van Horssen '14].
  Full characterisation in terms of non differentiability of Rényi entropy on $\mathbb{R}_+$[BJPP '16 in preparation].
  Link with selection of invariant states and the failure of the CLT[B., Pautrat, Pellegrini in preparation].
  First approach to the characterisation of metastable behaviour [Macieszczak, Guta, Lesanovsky, Garrahan '16].

Open questions

Markov Chain

- Regularity of the invariant measure $\nu_{inv.}$.
- Relaxation of **(Pur.)**. Mixing for products of i.i.d. elements of $SU(d)$?
- Large Deviation Principle for the chain. Existence of a spectral gap for Π.
- Entropy production of the chain and time reversibility.

Dynamical system

- Meaning of the irregularities of the Rényi entropy in terms of dynamical phases beyond non differentiability on $\mathbb{R}_+$.
- Full characterisation of Dynamical Phase Transitions and metastability.

A problem for continuous time quantum trajectories

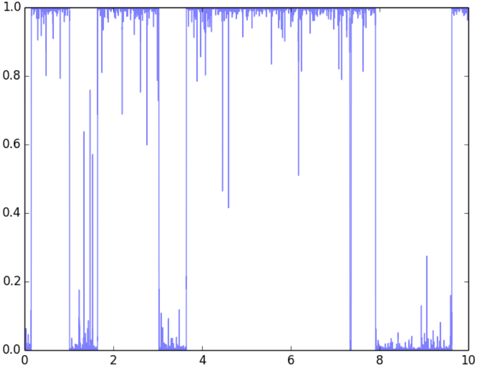
In the appropriate continuous time limit, discrete quantum trajectories converge weakly towards solutions of stochastic differential equations[Pellegrini '08-'10].

One particular issue arise in the emergence of quantum jumps[Bauer,Bernard,Tilloy '15].
The energy population p_t of the exited state of a two level atom being indirectly measured by a diffusive signal is a process solution of the following diffusive SDE.

$$dp_t = \lambda(\frac{1}{2} - p_t)dt + \sqrt{\gamma}p_t(1 - p_t)dW_t, \quad p_0 \in (0, 1).$$

What is the limit of $(p_t)_t$ when $\gamma \to \infty$?

Numerical simulation of $(p_t)_t$ $(\lambda = 1, \gamma = 10^4)$:



The "spikes" do not disappear when the simulation is refined.

How to make sense of the limit?

- Use an appropriate topology that does not see the "spikes" (Meyer–Zheng convergence in measure topology).
- Find a meaning full limit towards a random variable taking value in nowhere continuous functions.
- Limit towards a Poisson measure on $[0,1] \times [0,T]$ for the local maxima.

Another approach [Bauer, Bernard, Tilloy '16]:
Study in effective time by Dubins-Schwarz Theorem. Let

$$\tau_s := \inf \left\{ t : \gamma \int_0^t p_u^2 (1-p_u)^2 du > s \right\}.$$

Then $\tilde{p}_s := p_{\tau_s}$ should be the solution of

$$d\tilde{p}_s = \frac{\frac{1}{2} - \tilde{p}_s}{\gamma \tilde{p}_s^2 (1-\tilde{p}_s)^2} ds + dB_s, \quad \tilde{p}_0 = p_0.$$

[Bauer, Bernard, Tilloy '16] provides a formal proof that $(\tilde{p}_s)_s$ converges towards a Brownian motion reflected in 0 and 1.

Thank you!