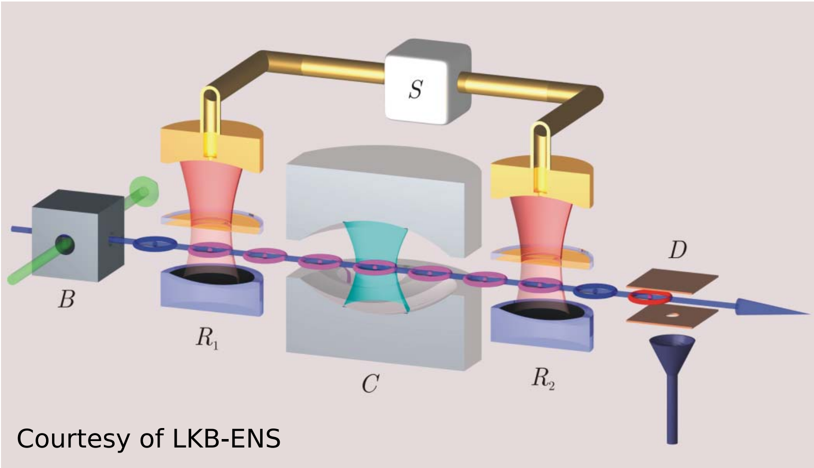
Repeated nondemolition measurements and wave function collapse

Open Quantum Systems summer school
Autrans July 2013

Tristan Benoist
PhD. student of Denis Bernard
Joint work with Michel Bauer and Denis Bernard

Serge Haroche's group experiment

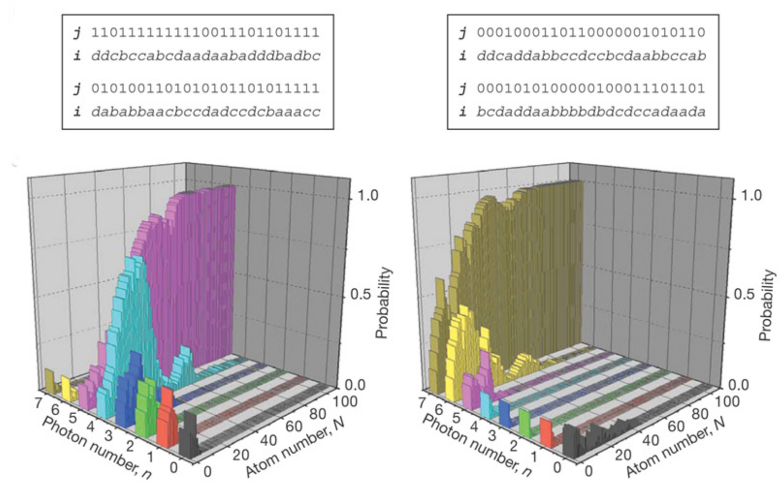


Courtesy of LKB-ENS

- Monochromatic photon field trapped in a superconducting cavity.
- Probed by several Rydberg atoms.

- Measurement on the atoms lead to a collapse of the field to a fixed photon number fock state.
- The limit fock states distribution corresponds to Von Neumann postulate for a direct measurement.

Ref.: C. Guerlin et al. Nature **448**, 889-893



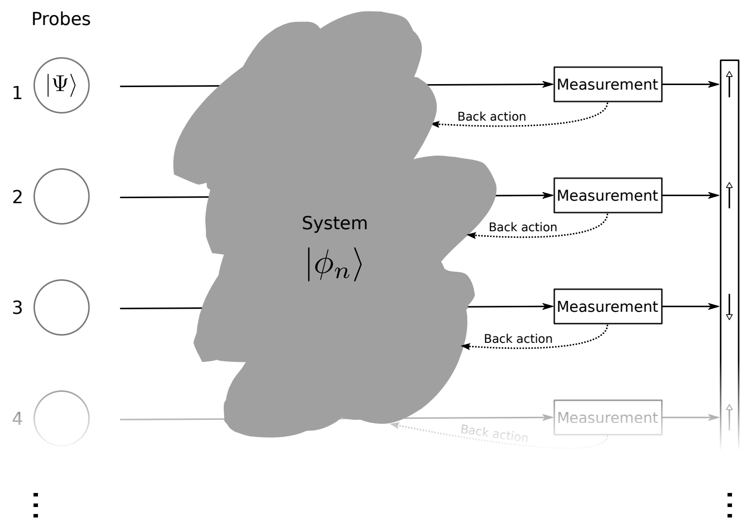
Courtesy of LKB-ENS

Questions :

- How the state collapse ?
- What is the non-degeneracy criteria ?
- How the final state distribution depends on the trial state ?
- What is the convergence speed ?
- What is the influence of the utilization of different probe/measurement types ?

What is a nondemolition measurement ?

Measurement scheme



System state update

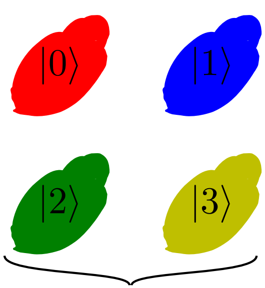
$$|\phi_{n+1}\rangle = \frac{1}{\|\langle \uparrow | U\Psi\rangle|\phi_n\rangle\|}\langle \uparrow | U\Psi\rangle|\phi_n\rangle$$

State of the system at time n

$$|\phi_n(\uparrow, \uparrow, \ldots, \downarrow)\rangle$$

QND = a system Hilbert space basis composed of stable states under the measurement process

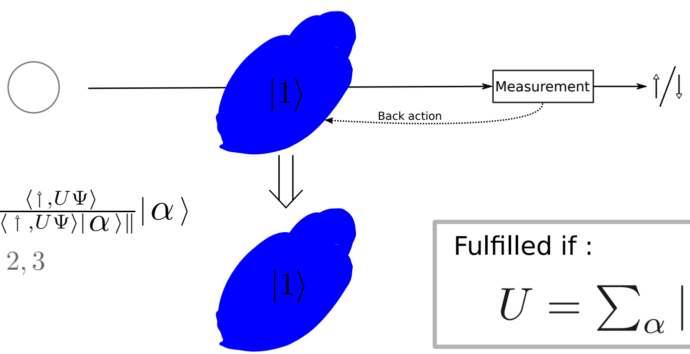
For a 4 level system :



Pointer states

such that : $|\alpha\rangle = \frac{\langle \uparrow, U\Psi\rangle}{\|\langle \uparrow, U\Psi\rangle|\alpha\rangle\|}|\alpha\rangle$

$\alpha = 0, 1, 2, 3$



Fulfilled if :

$$U = \sum_\alpha |\alpha\rangle\langle\alpha| \otimes U(\alpha)$$

Law of the measurement outcomes

- **If the initial system state is a pointer**

$$|\phi_0\rangle = |\alpha\rangle, \; |\phi_n\rangle = |\alpha\rangle$$

so

$$\|\langle\upharpoonleft, U\Psi\rangle|\phi_n\rangle\|^2 = |\langle\upharpoonleft, U(\alpha)\Psi\rangle|^2 = p(\upharpoonleft|\alpha)$$

Law of the measurement outcomes : $\mathbb{P}_\alpha(\upharpoonleft, \downharpoonleft, \ldots, \downharpoonleft) = p(\upharpoonleft|\alpha)p(\downharpoonleft|\alpha)\ldots p(\downharpoonleft|\alpha)$

To each pointer state we can associate a law for the outcomes : If $|\phi_0\rangle = |\alpha\rangle$ then, outcomes follow $\mathbb{P}_\alpha$

Under this law, the measurement outcomes are i.i.d. random variables.

- **If the initial state is not a pointer state**

$$|\phi_0\rangle \neq |0\rangle, |1\rangle, |2\rangle, |3\rangle$$

Law of the measurement outcomes :

$$\boxed{\mathbb{P}(\upharpoonleft, \downharpoonleft, \ldots, \downharpoonleft) = \sum_\alpha q_0(\alpha)\mathbb{P}_\alpha(\upharpoonleft, \downharpoonleft, \ldots, \downharpoonleft)}$$

Under this law, the measurement outcomes are no longer i.i.d. random variables.

$q_n(\alpha) = |\langle\alpha|\phi_n\rangle|^2$ Probability to obtain the pointer if the system is directly measured at time *n*.

Martingale change of measure and convergence

- **If $q_0(\alpha) > 0$, $\mathbb{P}_\alpha$ is absolutely continuous with respect to $\mathbb{P}$. So it exists a martingale $Q_n(\alpha)$ such that :**

$$q_0(\alpha)\mathbb{P}_\alpha(\underbrace{\upharpoonleft, \downharpoonleft, \dots, \downharpoonleft}_{n}) = Q_n(\alpha; \upharpoonleft, \downharpoonleft, \dots, \downharpoonleft)\mathbb{P}(\upharpoonleft, \downharpoonleft, \dots, \downharpoonleft)$$

$$Q_n(\alpha) = q_0(\alpha)\frac{\mathbb{P}_\alpha(\upharpoonleft, \downharpoonleft, \dots, \downharpoonleft)}{\mathbb{P}(\upharpoonleft, \downharpoonleft, \dots, \downharpoonleft)}$$

- **Evolution of $q_n(\alpha) = |\langle\alpha|\phi_n\rangle|^2$ if the measurement outcomes are : $\upharpoonleft, \downharpoonleft, \dots, \downharpoonleft$**

$$q_n(\alpha) = q_0(\alpha)\frac{\mathbb{P}_\alpha(\upharpoonleft, \downharpoonleft, \dots, \downharpoonleft)}{\mathbb{P}(\upharpoonleft, \downharpoonleft, \dots, \downharpoonleft)}$$

$$\boxed{q_n(\alpha) = Q_n(\alpha)}$$

$q_n(\alpha) = |\langle\alpha|\phi_n\rangle|^2$ Probability to obtain the pointer if the system is directly measured at time *n*.

- **Under $\mathbb{P}$ the "probability" of each pointer is a bounded martingale and therefore converges.**

$$\lim_{n\to\infty} q_n(\alpha) = q_\infty(\alpha) \qquad q_0(\alpha)\mathbb{P}_\alpha = q_\infty(\alpha)\mathbb{P}$$

Non degeneracy criteria

Limit of $q_n(\alpha)$ $\Rightarrow$ $q_\infty(\alpha)q_\infty(\beta)\left(p(\upharpoonleft|\alpha)-p(\upharpoonleft|\beta)\right)=0$

$$p(\cdot|\alpha)\neq p(\cdot|\beta)\Rightarrow q_\infty(\alpha)=\mathbf{1}_{\alpha=\Upsilon}$$

$\Upsilon=0,1,2,3$

Example $p(\cdot|\alpha)$

○	0	1	2	3
↓	3/4	1/3	2/3	1/4
↑	1/4	2/3	1/3	3/4

Υ is a random variable on the pointer state set.

$$q_\infty(\alpha)=\mathbf{1}_{\alpha=\Upsilon}\quad\Rightarrow\quad\boxed{\lim_{n\to\infty}|\phi_n\rangle=|\Upsilon\rangle}$$

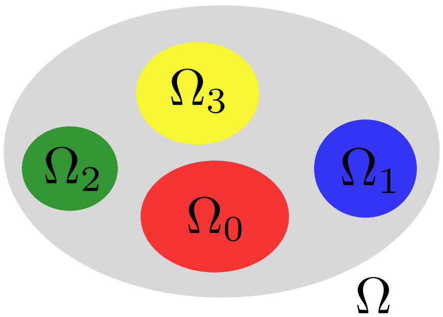
$$\mathbb{E}[q_\infty(\alpha)]=q_0(\alpha)\quad\Rightarrow\quad\boxed{\mathbb{P}(\Upsilon=\alpha)=|\langle\alpha|\phi_0\rangle|^2}$$

$\Rightarrow$ **In the large n limit, the system state behaves as if a projective Von Neumann measurement would have been performed at time 0.**

Realization space partition and non degeneracy

The non degeneracy is implied by the $\mathbb{P}_\alpha$'s beeing mutually singular.

$$\boxed{\mathbb{P}_\alpha(\Omega_\beta) = \delta_{\alpha,\beta} \Rightarrow q_\infty(\alpha) = \mathbf{1}_{\alpha=\Upsilon}}$$



$\mathbb{P}(\cup_\alpha \Omega_\alpha) = 1$ and $\Omega_\alpha \cap \Omega_\beta \underset{\alpha\neq\beta}{=} \emptyset$

In our example :

$$\Omega_0 = \{\omega \in \Omega \text{ s.t. } \lim_n \frac{N_n(\cdot;\omega)}{n} = p(\cdot|0)\}$$

$$q_0(\alpha)\mathbb{P}_\alpha(\Omega_\beta) \underset{\alpha\neq\beta}{=} 0 = \mathbb{E}[q_\infty(\alpha)\mathbf{1}_{\Omega_\beta}] \quad \Rightarrow \quad q_\infty(\alpha)\mathbf{1}_{\Omega_\beta} = 0,\ \mathbb{P}-a.s.$$

$$\sum_\alpha q_\infty(\alpha) = 1 \quad \Rightarrow \quad q_\infty(\beta) = \mathbf{1}_{\Omega_\beta},\ \mathbb{P}-a.s.$$

$\Upsilon = \sum_\alpha \alpha \mathbf{1}_{\Omega_\alpha}$ and $\mathbb{P}(\Upsilon = \alpha) = \mathbb{P}(\Omega_\alpha) = q_0(\alpha)$

$$q_\infty(\alpha) = \mathbf{1}_{\alpha=\Upsilon} \quad \Rightarrow \quad \boxed{\lim_{n\to\infty} |\phi_n\rangle = |\Upsilon\rangle}$$

$\Rightarrow$ **In the large n limit, the system state behaves as if a projective Von Neumann measurement would have been performed at time 0.**

Remark : **The limit state does not depend on the initial system state.
We know the limit state just by looking at the measurement results without computing the system state evolution.**

Exponential convergence

- $(\uparrow, \downarrow, \dots, \downarrow)$ is a sequence of i.i.d. random variables under $\mathbb{P}_\alpha$

  We can apply the law of large numbers :

$$N_n(\cdot)/n \xrightarrow[n\to\infty]{} p(\cdot|\alpha),\ \mathbb{P}_\alpha - a.s. \qquad \Rightarrow \qquad \boxed{N_n(\cdot)/n \xrightarrow[n\to\infty]{} p(\cdot|\Upsilon),\ \mathbb{P} - a.s.}$$

- Recall : $q_n(\alpha) = q_0(\alpha)\frac{p(\uparrow|\alpha)^{N_n(\uparrow)}p(\downarrow|\alpha)^{N_n(\downarrow)}}{Z_n}$ with $Z_n = \sum_\beta q_0(\beta)p(\uparrow|\beta)^{N_n(\uparrow)}p(\downarrow|\beta)^{N_n(\downarrow)}$

  Applying L.L.N. :

$$Z_n \simeq e^{nS(\Upsilon)}$$

$$S(\Upsilon) = p(\downarrow|\Upsilon)\ln[p(\downarrow|\Upsilon)] + p(\uparrow|\Upsilon)\ln[p(\uparrow|\Upsilon)]$$

$$\Downarrow$$

$$\boxed{q_n(\alpha) \simeq e^{-nS(\Upsilon|\alpha)}}$$

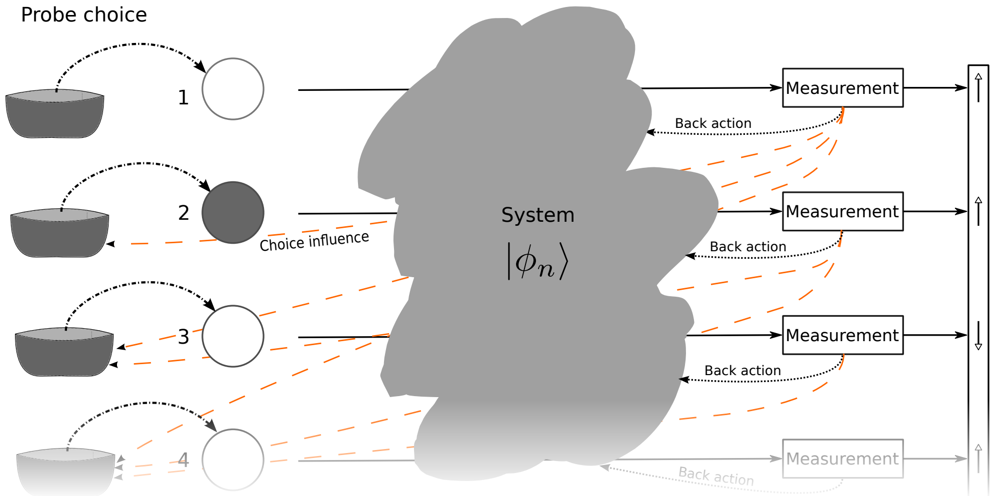
$S(\Upsilon|\alpha)$ relative entropy of measurement result distribution knowing Υ with respect to the measurement result distribution knowing α

$$\boxed{S(\Upsilon|\alpha) = \sum_j p(j|\Upsilon)\ln\left[\frac{p(j|\Upsilon)}{p(j|\alpha)}\right]}$$

$j = \downarrow, \uparrow$

$$S(\gamma|\alpha) \geq 0 \text{ and } S(\gamma|\alpha) = 0 \text{ i.f.f. } \ p(\cdot|\alpha) = p(\cdot|\gamma)$$

Using different probes



System state update

$$|\phi_{n+1}\rangle = \frac{\langle \uparrow, U \bullet \rangle}{\| \langle \uparrow, U \bullet \rangle |\phi_n\rangle \|} |\phi_n\rangle$$

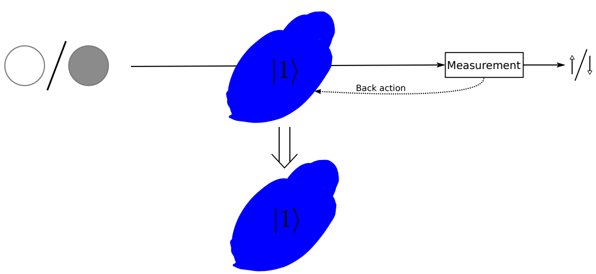
State of the system at time n

$$|\phi_n(\circ, \uparrow, \bullet, \uparrow, \ldots, \bullet, \downarrow)\rangle$$

The way probes are chosen can depend on time and all history :

$$c_n(\circ | \circ, \uparrow, \ldots, \bullet, \downarrow)$$

The pointers are still stable which ever is the probe :



In S. Haroche's group experiment :
4 different types of indirect measurements chose randomly at each time

- **Do we still have convergence ?**
- **What is the non degeneracy criteria ?**
- **What is the convergence rate ?**

Martingale change of measure and convergence

We change the realizations and the measure but we keep the main features.

- **Measurement realization:** $\circ, \uparrow, \bullet, \uparrow, \dots$

- **Measures:**

If the initial state is a pointer state: $|\phi_0\rangle = |\alpha\rangle$

$$\mathbb{P}_\alpha(\underbrace{\circ, \uparrow, \bullet, \dots, \circ, \uparrow, \bullet, \downarrow}_{2n+2}) = c_0(\circ)c_1(\bullet|\circ, \uparrow) \dots c_n(\bullet|\circ, \uparrow, \dots, \circ, \uparrow)p^{\circ}(\uparrow|\alpha) \dots p^{\bullet}(\downarrow|\alpha)$$

If the initial state is not one of the pointer state: $|\phi_0\rangle \neq |\alpha\rangle$

$$\mathbb{P} = \sum_\alpha q_0(\alpha)\mathbb{P}_\alpha$$

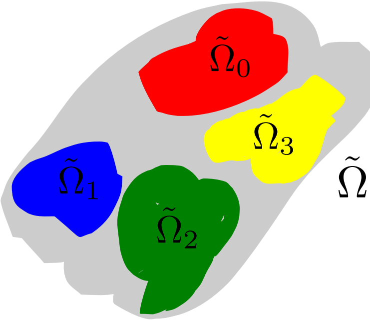
We still have a martingale $\boxed{q_n(\alpha) = q_0(\alpha)\frac{\mathbb{P}_\alpha(\circ, \uparrow, \dots, \circ, \uparrow)}{\mathbb{P}(\circ, \uparrow, \dots, \circ, \uparrow)}}$

With $q_n(\alpha) = |\langle\alpha|\phi_n\rangle|^2$

Convergence and non degeneracy criteria

$$q_n(\alpha) = q_0(\alpha)\frac{p^{\circ}(\uparrow|\alpha)\dots p^{\bullet}(\downarrow|\alpha)}{\sum_\beta q_0(\beta)p^{\circ}(\uparrow|\beta)\dots p^{\bullet}(\downarrow|\beta)}$$ still is a bounded martingale under $\mathbb{P}$

As before, if the measures corresponding to each pointers are mutually singular, we have projection of the wave function.



$$\mathbb{P}_\alpha(\tilde{\Omega}_\beta) = \delta_{\alpha,\beta} \;\Rightarrow\; q_\infty(\alpha) = \mathbf{1}_{\tilde{\Omega}_\alpha}$$

Some conditions under which this condition is fulfilled :

If for any two pointers **we use almost surely infinitely many times** a measurement type fulfilling the previous non degeneracy condition, then the wave function collapse as in a direct von Neumann measurement of the pointer states;

$$\boxed{\left.\begin{array}{l}\lim_n N_n(\bullet) = +\infty \\ p^{\bullet}(\cdot|\alpha) \neq p^{\bullet}(\cdot|\beta)\end{array}\right\} \;\Rightarrow\; q_\infty(\alpha)q_\infty(\beta) = 0} \quad \Rightarrow \quad \boxed{\begin{array}{l}\lim_{n\to\infty}|\phi_n\rangle = |\Upsilon\rangle \\ \mathbb{P}(\Upsilon = \alpha) = |\langle\alpha|\phi_0\rangle|^2\end{array}}$$

$\Rightarrow$ **In the large n limit, the system state behaves as if a projective Von Neumann measurement would have been performed at time 0.**

Remark : **The limit state does not depend on the initial system state.**
We know the limit state just by looking at the measurement results without computing the system state evolution.

Markovian choice of probe and exponential convergence

The way one choose the probe depends only on the preceeding probe and measurement result.

$$c_n(\circ|\circ, \uparrow, \dots, \bullet, \downarrow) = c(\circ|\bullet, \downarrow)$$

$\circ, \uparrow, \bullet, \uparrow, \dots$ is a Markov chain

If the initial state is a pointer state, its transition kernel is:

$$K_\alpha(\bullet, \downarrow|\circ, \uparrow) = c(\circ|\bullet, \downarrow)p^{\circ}(\uparrow|\alpha)$$

If this kernel is irreductible and aperiodic, the law of large numbers gives us:

$$\frac{N_n(\bullet, \uparrow)}{n} \xrightarrow[n\to\infty]{\mathbb{P}_\alpha - a.s.} \mu_\alpha(\bullet)p^{\bullet}(\uparrow|\alpha)$$

Again we have an exponential convergence

$$\boxed{\begin{array}{l} q_n(\alpha) \simeq e^{-n\overline{S}(\Upsilon|\alpha)} \\ \overline{S}(\Upsilon|\alpha) = \sum_o \mu_\Upsilon(o) S^o(\Upsilon|\alpha) \end{array}}$$

$o = \bullet, \circ$

The rate is a mean of the different possible rate with a distribution which can depend on the limit pointer state.

Impact of probe choice freedom

This choice freedom allows to increase the convergence rate towards the limit state.

$$\min_{o=\bullet,\circ} S^o(\gamma|\alpha) \leq \overline{S}(\gamma|\alpha) \leq \max_{o=\bullet,\circ} S^o(\gamma|\alpha)$$

Relative entropies :

○	0	1	2	3
0	0	0.38	0.23	0.017
1	0.36	0	0.016	0.55
2	0.23	0.017	0	0.38
3	0.016	0.55	0.36	0

minimum

●	0	1	2	3
0	0	0.36	0.016	0.55
1	0.38	0	0.23	0.017
2	0.017	0.23	0	0.38
3	0.55	0.017	0.36	0

minimum

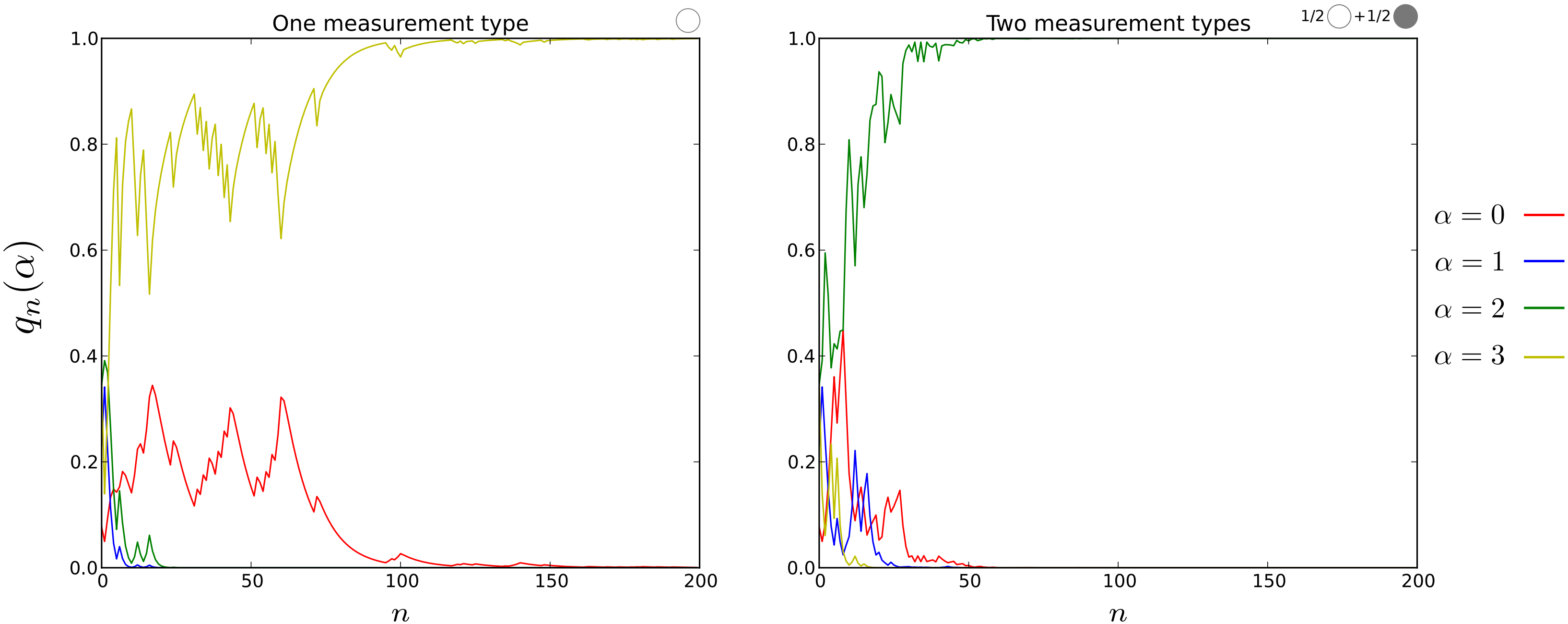
○ ●	0	1	2	3
0	0	0.37	0.12	0.28
1	0.37	0	0.12	0.28
2	0.12	0.12	0	0.38
3	0.28	0.28	0.36	0

$c(\cdot) = 1/2$

minimum

	$p^{\cdot}(\cdot\|\alpha)$	0	1	2	3
○	↓	3/4	1/3	2/3	1/4
	↑	1/4	2/3	1/3	3/4
●	↓	2/3	1/4	1/3	3/4
	↑	1/3	3/4	1/3	1/4

Impact of probe choice freedom



Simulations

Trial state Independence

In S. Haroche's experiment, the *true* state of the system is unknown.
The computations are made using a *trial* state.

- True state : $|\phi_0\rangle$ Trial state : $|\hat{\phi}_0\rangle = \frac{1}{\sqrt{4}}(|0\rangle + |1\rangle + |2\rangle + |3\rangle)$

$$|\hat{\phi}_{n+1}\rangle = \frac{\langle \uparrow | U \bigcirc \rangle}{\| \langle \uparrow | U \bigcirc \rangle |\hat{\phi}_n\rangle \|} |\hat{\phi}_n\rangle \qquad \hat{q}_n(\alpha) = |\langle \alpha | \hat{\phi}_n \rangle|^2$$

- The law of the outcomes still is $\mathbb{P} = \sum_\alpha q_0(\alpha)\mathbb{P}_\alpha$

  $\hat{q}_n(\alpha)$ is not a martingale under $\mathbb{P}$ but it is under $\hat{\mathbb{P}} = \sum_\alpha \hat{q}_0(\alpha)\mathbb{P}_\alpha$ We consider the trial state as the true state.

- Repeating our study under this *trial* measure, we obtain the convergence of $\hat{q}_n(\alpha)$

$$\hat{q}_n(\alpha) \xrightarrow[n\to\infty]{\hat{\mathbb{P}}-a.s} \mathbf{1}_{\alpha=\Upsilon}$$

- If $\hat{q}_0(\alpha) > 0$ for any α such that $q_0(\alpha) > 0$, then $\hat{\mathbb{P}}(A) = 1 \Rightarrow \mathbb{P}(A) = 1$

So $\hat{q}_n(\alpha) \xrightarrow[n\to\infty]{\mathbb{P}-a.s} \mathbf{1}_{\alpha=\Upsilon}$

Υ does not depend on the initial state.
Only its law depends on it.

$$\boxed{|\hat{\phi}_n\rangle \xrightarrow[n\to\infty]{\mathbb{P}-a.s.} |\Upsilon\rangle}$$

The limit trial state is the same as the limit true state.

Continuous limit - outcome processes (finite dimensional joint distributions limit)

$$U = e^{-i\delta(H_{sys.}\otimes\mathbb{I}+\mathbb{I}\otimes H_{probe}+\frac{1}{\sqrt{\delta}}H_{int.})}$$

δ interaction time duration

Continuous time limit : $\lim \delta \to 0,\ n\delta = t$ fixed.

We always use the same probe.

- **From the counting processes $N_n(j)$ we define processes $Y_t^\delta(j)$, search for their continuous limit and the law of their continuous limit.** All the information on the outcomes is contained in these processes.
- **Difficulty:** The measures $\mathbb{P}_\alpha$ depend on δ
- **Limit derivation scheme:**

Finite dimensional characteristic functions

Of $Y_t^\delta(j)$ under $\mathbb{P}_\alpha$

Of some cannonical processes $X_t(j)$ under a measure μ_0

Continuous limit

Girsanov transformation

Martingale $M_t(\alpha)$

$\mu_\alpha = M_t(\alpha)\mu_0$

$\mathcal{F}^{\mathbf{Y}}_\alpha$

$\mathbb{P}_\alpha \to \mathbb{P}$

Weighted sum over the pointers

Girsanov transformation

Martingale M_t

$M_t = \sum_\alpha q_0(\alpha) M_t(\alpha)$

$\mu = M_t\mu_0$

$\mathcal{F}^{\mathbf{Y}}$

In the continuous limit the finite dimensional characteristic function of $Y_t^\delta(j)$ under $\mathbb{P}_\square$ is equal to the one of $X_t(j)$ under $\mu_\square$. $\square$ standing either for α or nothing.

Continuous limit - pointer "probability" (finite dimensional joint distributions limit)

- **As in the discrete case, the measures μ_α are absolutely continuous with respect to μ**

  We have a martingale such that:

$$q_0(\alpha)\mu_\alpha = \tilde{Q}_t(\alpha)\mu \quad \text{and} \quad \tilde{Q}_t(\alpha) = q_0(\alpha)\frac{M_t(\alpha)}{M_t}$$

- **We search the continuous limit of $q_n(\alpha)$**

1- The $q_{[t/\delta]}(\,\cdot\,)$'s are functions of the $Y_t^\delta(j)$'s. We can find the continuous limit of their joint finite dimensional characteristic function under $\mathbb{P}_\alpha$. (Actually we compute limit of Mellin transforms)

2- We identify it with the joint finite dimensional characteristic function of processes $\tilde{q}_t(\cdot)$ under μ_α.

3- Using the definition of $\mathbb{P}$ and μ we have the same identification between the continuous limit of the characteristic function under $\mathbb{P}$ and the characteristic function under μ .

The martingale change of measure gives us the law of $\tilde{q}_t(\cdot)$ under μ.

$$\boxed{q_{[t/\delta]}(\alpha) \xrightarrow[\delta\to 0]{\mathbb{P}\to\mu} \tilde{q}_t(\alpha)}$$

- **Once again, we can identify the limit continuous process with the Radon-Nikodym of μ_α with respect to μ.**

Under $\mu_\square$, $\tilde{q}_t(\alpha)$ has the same law as $\tilde{Q}_t(\alpha)$.

Thank you for your attention.

Degeneracy

Degeneracy appear when the measures corresponding to pointer states are not mutually singular.

$$p(\cdot|\alpha) = p(\cdot|\beta)$$

$$\Downarrow$$

$$q_\infty(\alpha) + q_\infty(\beta) = \begin{cases} 1 & \text{with prob. } q_0(\alpha) + q_0(\beta) \\ 0 & \text{else.} \end{cases}$$

$$q_\infty(\alpha) = \begin{cases} \frac{q_0(\alpha)}{q_0(\alpha)+q_0(\beta)} & \text{with prob. } q_0(\alpha) + q_0(\beta) \\ 0 & \text{else.} \end{cases}$$

Equivalent to a Von Neuman initial measurement.

The state does not automatically converge.

$$\langle \uparrow | U(\alpha) \bigcirc \rangle = e^{-i\theta(\uparrow|\alpha)} \sqrt{p(\uparrow|\alpha)}$$

If the phase is compensated, we find a convergence.

$$\tilde{U}_n = \sum_\alpha e^{-i(\theta(\uparrow|\alpha)N_n(\uparrow) + \theta(\downarrow|\alpha)N_n(\downarrow))} |\alpha\rangle\langle\alpha|$$

$$\boxed{\tilde{U}_n^* |\phi_n\rangle \xrightarrow[n\to\infty]{\mathbb{P}-a.s.} |\phi_\infty\rangle}$$

$|\phi_\infty\rangle$ corresponds to the random state resulting from an initial direct von Neumann measurement.