Vanishing of entropy production and quantum detailed balance

joint work with N. Cuneo, V. Jakšić, Y. Pautrat, C.-A. Pillet

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MCA, July 2021

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Equilibrium vs. Out of equilibrium

How can we characterize equilibrium?

Second law

Macroscopic Irreversibility:

Clausius (1850), thermodynamic:

$$Ep := \Delta S \geq 0,$$

$$\textbf{Equilibrium} \implies Ep = 0.$$

Do we have the converse?

For classical Markov chains:

$$Ep = 0 \text{ iff the process is } \textbf{reversible} \text{ iff } \textbf{detailed balance} \text{ holds.}$$

Definition (Detailed balance)

A Markov kernel P with invariant measure μ verifies detailed balance condition if P is self adjoint with respect to the inner product

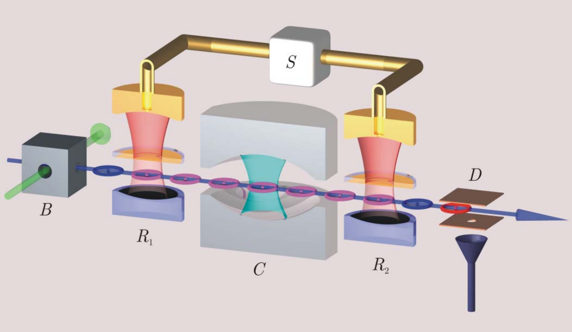
$$\langle f, g \rangle_\mu = \int \bar{f} g \, d\mu.$$

Do we have equivalence for quantum channels?

First, we have to define notions of detailed balance and entropy production.

Second, we have to show some equivalence.

S. Haroche group experiment:



j	1101111111110011101101111
i	*ddcbccabcdaadaabadddbadbc*
j	0101001101010101101011111
i	*dababbaacbccdadccdcbaaacc*

j	0001000110110000001010110
i	*ddcaddabbccdccbcdaabbccab*
j	0001010100000100011101101
i	*bcdaddaabbbbdbdcdccadaada*

Image: LKB ENS

States and quantum channels

Definition (States)
A state ρ is a trace 1 positive semi definite matrix:

$$\rho \in \mathcal{D} := \{\mu \in M_d(\mathbb{C}) \mid \mu \geq 0,\ \operatorname{tr} \mu = 1\}.$$

Definition (Completely Positive (CP) maps)
A positive linear map $\Psi : M_d(\mathbb{C}) \to M_d(\mathbb{C})$ is said completely positive if $\Psi \otimes \mathrm{Id}_{M_n(\mathbb{C})}$ is positive for any $n \in \mathbb{N}$.

Definition (Quantum channels)
A quantum channel is a CP map Φ that preserves the identity: $\Phi(I) = I$. Equivalently $\operatorname{tr} \circ \Phi^* = \operatorname{tr}$.

Average evolution for repeated interactions:

$$\bar{\rho}_{n+1} := \Phi^*(\bar{\rho}_n).$$

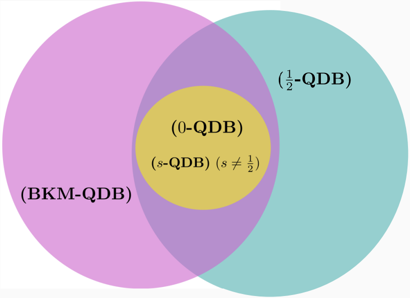
Quantum detailed balance

Definition (Quantum detailed balance)

For any $s \in \mathbb{R}$, let $\Phi^{(s)}$ be the adjoint of Φ w.r.t. the scalar product $\langle A, B \rangle_s = \mathrm{tr}(\rho^s A^* \rho^{1-s} B)$.

Then the quantum channel Φ verifies (anti)unitary **(s-QDB)** if there exists an (anti)unitary operator $J : \mathbb{C}^d \to \mathbb{C}^d$ such that, there exists $c \in U(1)$ such that $\Phi(J^2) = cJ^2$ and

$$\Phi^{(s)} = J^* \Phi(J \cdot J^*) J.$$



Reminder: $\langle \cdot, \cdot \rangle_{BKM} = \int_0^1 \langle \cdot, \cdot \rangle_s ds$.

Mainly follows from [Fagnola, Umanità '08 and Carlen, Maas '17].

Instruments

Definition (Instruments)
A quantum instrument,

$$\mathcal{J} := \{\Phi_a : M_d(\mathbb{C}) \to M_d(\mathbb{C})\}_{a\in\mathcal{A}}$$

is a set of CP maps such that

$$\Phi := \sum_{a\in\mathcal{A}} \Phi_a$$

is a quantum channel.

Meaning:
The letter a summarizes the measurement result after one interaction and Φ_a encodes the effect of the interaction, given the measurement result is a.

After one indirect measurement,

$$\rho_1 := \frac{\Phi_a^*(\rho_0)}{\operatorname{tr}\Phi_a^*(\rho_0)} \text{ with prob. } \operatorname{tr}\Phi_a^*(\rho_0).$$

Probability measures

Definition
Let $\mathcal{J}$ be a quantum instrument. Let $\rho \in \mathcal{D}$. Then the probability to measure the finite sequence $a_1, a_2, \ldots, a_n$ is,

$$\mathbb{P}(a_1, \ldots, a_n) = \mathrm{tr}(\rho \Phi_{a_1} \circ \cdots \circ \Phi_{a_n}(I_d)).$$

Summary: Such $\mathbb{P}$ is the distribution of the data sequence obtained in a repeated quantum measurement experiment.

Probability measure properties

Lemma (Ergodic property)
If Φ *is* **irreducible**, *then* $\mathbb{P}$ *(defined by* ρ*) is ergodic w.r.t. the left shift.*

Lemma (Upper quasi Bernoulli property)
If $\rho > 0$, *then* $\exists C > 0$ *such that, for any two finite sequences* $a_1, \ldots, a_n$ *and* $b_1, \ldots, b_p$,

$$\mathbb{P}(a_1, \ldots, a_n, b_1, \ldots, b_p) \leq C\mathbb{P}(a_1, \ldots, a_n)\mathbb{P}(b_1, \ldots, b_p).$$

From now on, I assume Φ is irreducible and ρ is the unique invariant state of Φ^*.

Order reversal

Reversal of the movie frames.

Let θ be an involution of $\mathcal{A}$. For example $\theta(1) = 2,\ \theta(2) = 1 \dots$

A time reversal of the data sequence is then:

$$\Theta_n(a_1, \dots, a_n) := (\theta(a_n), \dots, \theta(a_1)).$$

The probability of the reversed sequence is:

$$\widehat{\mathbb{P}}(a_1, \dots, a_n) = \mathbb{P}(\theta(a_n), \dots, \theta(a_1)).$$

Canonical order reversal instrument

$\widehat{\mathbb{P}}$ is always the unraveling of a reversed $\widehat{\Phi}$ by a reversed instrument $\widehat{\mathcal{J}} := \{\widehat{\Phi}_a\}_a$.

An example is:

$$\widehat{\Phi}_a(X) = J^* \rho^{-\frac{1}{2}} \Phi^*_{\theta(a)}(\rho^{\frac{1}{2}} J X J^* \rho^{\frac{1}{2}}) \rho^{-\frac{1}{2}} J$$

with $J : M_d(\mathbb{C}) \to M_d(\mathbb{C})$ an (anti)unitary operator such that $[J, \rho] = 0$.

Remark that $\widehat{\Phi}$ is the dual of $J\Phi(J^* \cdot J)J^*$ w.r.t. the inner product $\langle \cdot, \cdot \rangle_{\frac{1}{2}}$.

Relative entropy convergence

The entropy production is

$$S_n(\mathbb{P}|\widehat{\mathbb{P}}) := \sum_{a_1,\dots,a_n} \mathbb{P}(a_1,\dots,a_n) \log \frac{\mathbb{P}(a_1,\dots,a_n)}{\widehat{\mathbb{P}}(a_1,\dots,a_n)} \geq 0.$$

Theorem (B., Jakšić, Pautrat, Pillet '16)
Assume Φ is irreducible. Then,

$$Ep := \lim_{n\to\infty} \frac{1}{n} S_n(\mathbb{P}|\widehat{\mathbb{P}})$$

exists and

$$Ep = 0 \Leftrightarrow \mathbb{P} = \widehat{\mathbb{P}}.$$

Entropy production vanishes iff the measurement outcome process is reversible.

What is the relation between $\mathbb{P} = \widehat{\mathbb{P}}$ and **($\frac{1}{2}$-QDB)**?

Stinespring's dilation

Theorem (Stinespring's dilation theorem '55)
If Φ is a quantum channel, there exists $k \in \mathbb{N}$ and an isometry $V : \mathbb{C}^d \to \mathbb{C}^d \otimes \mathbb{C}^k$ such that

$$\Phi(X) = V^*(X \otimes I_k)V \quad \forall X \in M_d(\mathbb{C}).$$

Moreover if V and W are two Stinespring dilations of the same quantum channel Φ, then there exists a unitary matrix $U \in M_k(\mathbb{C})$ such that

$$W = (I_d \otimes U)\ V.$$

Informationally complete instruments

Proposition
Given a dilation V of Φ, then for any instrument $\mathcal{J}$ that sums to Φ, there exists a POVM $\{M_a\}_{a\in\mathcal{A}}$ such that for any $a \in \mathcal{A}$,

$$\Phi_a(X) = V^*(X \otimes M_a)V \quad \forall X \in M_d(\mathbb{C}).$$

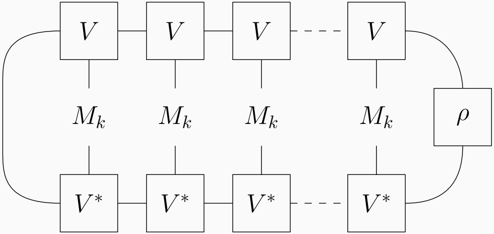
Definition (Informationally complete POVM)
The POVM $\{M_a\}_{a\in\mathcal{A}}$ is said informationally complete if $\operatorname{linspan}\{M_a\}_{a\in\mathcal{A}} = M_k(\mathbb{C})$.

Definition (Informationally complete instrument)
The instrument $\mathcal{J}$ is informationally complete if there exists an informationally complete POVM $\{M_a\}_{a\in\mathcal{A}}$ that can generate the instrument $\mathcal{J}$.

Finitely correlated state (FCS)

If $\mathcal{J}$ is an informationally complete instrument, $\mathbb{P} \equiv \omega$ with ω a purely generated FCS on $\bigotimes_{n\in\mathbb{Z}} M_k(\mathbb{C})$.

$\mathbb{P} \equiv \omega =$



If $A = \sum_{a_1,\dots,a_n} c_{a_1,\dots,a_n} M_{a_1} \otimes \cdots \otimes M_{a_n}$, $\omega(A) = \mathbb{E}(X_A)$ with $X_A = \sum_{a_1,\dots a_n} c_{a_1,\dots,a_n} \mathbf{1}_{a_1,\dots,a_n}$.

(Anti)Unitary implementable involution

Definition ((Anti)Unitarily implementable involution)
For a given POVM $\{M_a\}_{a\in\mathcal{A}}$, we say that the local involution θ is (anti)unitarily implementable if there exists a **(anti)unitary operator** $U \in M_k(\mathbb{C})$, such that

$$U^* M_a U = M_{\theta(a)}.$$

Theorem (B., Cuneo, Jakšić, Pautrat, Pillet '19)
Assume Φ is irreducible. Then, the following are equivalent.

- Φ *verifies (anti)unitary* **($\frac{1}{2}$-QDB)***;*
- *There exists an informationally complete instrument $\mathcal{J}$ summing to Φ and an (anti)unitarily implementable local involution θ such that $Ep = 0$.*

Proof:
$\Rightarrow$ Stinespring's dilation theorem and the duality operation is an involution.
$\Leftarrow$ [Fannes, Nachtergaele, Werner JFA '94; Guta, Kiukas '15] uniqueness of purely generated FCS. $\square$

Thank you!