# Vanishing of entropy production and quantum detailed balance

joint work with N. Cuneo, V. Jakšić, Y. Pautrat, C.-A. Pillet

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# Equilibrium vs. Out of equilibrium


# Second law

**Macroscopic Irreversibility:**
Clausius (1850), thermodynamic:

$$Ep := \Delta S \geq 0,$$

$$\textbf{Equilibrium} \implies Ep = 0.$$

**Do we have the converse ?**

For classical Markov chains, $Ep = 0$ iff **detailed balance** holds.

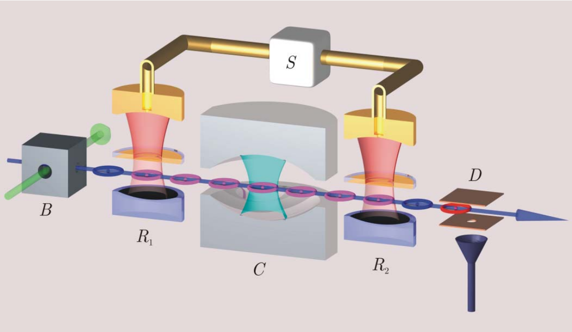
**Definition (Detailed balance)**
A Markov kernel $P$ with invariant measure $\mu$ verifies detailed balance condition if $P$ is self adjoint with respect to the inner product

$$\langle f, g \rangle_\mu = \int \bar{f} g \, d\mu.$$

Do we have equivalence for quantum repeated measurements?

# A canonical experiment

S. Haroche group experiment:




| | |
|---|---|
| **j** | 11011111111100111011011111 |
| **i** | *ddcbccabcdaadaabadddbadbc* |
| **j** | 01010011010101011010111111 |
| **i** | *dababbaacbccdadccdcbaaacc* |

| | |
|---|---|
| **j** | 00010001101100000010101110 |
| **i** | *ddcaddabbccdccbcdaabbccab* |
| **j** | 00010101000001000111011101 |
| **i** | *bcdaddaabbbbdbdcdccadaada* |

Image: LKB ENS

**Definition (States)**

A state $\rho$ is a trace 1 positive semi definite matrix:

$$\rho \in \mathcal{D} := \{\mu \in M_d(\mathbb{C}) \mid \mu \geq 0,\ \operatorname{tr} \mu = 1\}.$$

**Definition (Completely Positive (CP) maps)**

A positive linear map $\Psi : M_d(\mathbb{C}) \to M_d(\mathbb{C})$ is said completely positive iff. $\Psi \otimes \mathrm{Id}_{M_n(\mathbb{C})}$ is positive for any $n \in \mathbb{N}$.

**Definition (Quantum channels)**

A quantum channel is a CP map $\Phi$ that preserves the identity: $\Phi(I) = I$. Equivalently $\operatorname{tr} \circ \Phi^* = \operatorname{tr}$.

**Average evolution for repeated interactions:**

$$\bar{\rho}_{n+1} := \Phi^*(\bar{\rho}_n).$$

# Quantum detailed balance

**Definition (Quantum detailed balance)**

For any $s \in \mathbb{R}$, let $\Phi^{(s)}$ be the adjoint of $\Phi$ w.r.t. the scalar product $\langle A, B\rangle_s = \mathrm{tr}(\rho^s A^* \rho^{1-s} B)$.

Then the quantum channel $\Phi$ verifies **($s$-QDB)** if there exists an antiunitary involution $J : \mathbb{C}^d \to \mathbb{C}^d$ such that $\Phi^{(s)} = J\Phi(J \cdot J)J$.

**Remarks** (from [Fagnola, Umanità '08])

- $\Phi$ verifies **($s$-QDB)** for $s \neq \frac{1}{2}$ if and only if it verifies **(0-QDB)**;
- **(0-QDB)** $\Longrightarrow$ **($\frac{1}{2}$-QDB)** but **($\frac{1}{2}$-QDB)** $\not\Longrightarrow$ **(0-QDB)**;
- if $\Phi^{(0)}$ is a quantum channel, then for any $s \in \mathbb{R}$, $\Phi^{(s)} = \Phi^{(0)}$;
- **(0-QDB)** $\Longrightarrow$ $\Phi$ (restricted to a subset of $\mathcal{D}$) essentially represents the kernel of a classical Markov chain on $d$ states;
- the canonical reversal $\widehat{\Phi}$ is the $s = 1/2$ dual of $J\Phi(J \cdot J)J$.

**Definition (Instruments)**

A quantum instrument,

$$\mathcal{J} := \{\Phi_a : M_d(\mathbb{C}) \to M_d(\mathbb{C})\}_{a \in \mathcal{A}}$$

is a set of CP maps such that

$$\Phi := \sum_{a \in \mathcal{A}} \Phi_a$$

is a quantum channel.

**Meaning:**

The letter $a$ summarizes the measurement result after one interaction and $\Phi_a$ encodes the effect of the interaction, given the measurement result is $a$.

After one indirect measurement,

$$\rho_1 := \frac{\Phi_a^*(\rho_0)}{\operatorname{tr} \Phi_a^*(\rho_0)} \text{ with prob. } \operatorname{tr} \Phi_a^*(\rho_0).$$

**Definition**

Let $\mathcal{J}$ be a quantum instrument. Let $\rho \in \mathcal{D}$. Then the probability to measure the finite sequence $a_1, a_2, \ldots, a_n$ is,

$$\mathbb{P}(a_1, \ldots, a_n) = \mathrm{tr}(\rho \Phi_{a_1} \circ \cdots \circ \Phi_{a_n}(I_d)).$$

**Summary:** Such $\mathbb{P}$ is the distribution of the data sequence obtained in a repeated quantum measurement experiment.

# Probability measure properties

Ergodic property:

- If $\Phi$ is **irreducible** (*i.e.* $\Phi^*$ has a unique faithful stationary state $\rho = \Phi^*(\rho)$), then $\mathbb{P}$ is ergodic w.r.t. the left shift.

Upper quasi Bernoulli property:

- $\exists C > 0$ such that, for any two finite sequences $a_1, \ldots, a_n$ and $b_1, \ldots, b_p$,

$$\mathbb{P}(a_1, \ldots, a_n, b_1, \ldots, b_p) \leq C\mathbb{P}(a_1, \ldots, a_n)\mathbb{P}(b_1, \ldots, b_p).$$

# Order reversal

Let $\theta$ be an involution of $\mathcal{A}$. For example $\theta(1) = 2,\ \theta(2) = 1\ldots$

A time reversal of the data sequence is then:

$$\Theta_n(a_1, \ldots, a_n) := (\theta(a_n), \ldots, \theta(a_1)).$$

The probability of the reversed sequence is:

$$\widehat{\mathbb{P}}(a_1, \ldots, a_n) = \mathbb{P}(\theta(a_n), \ldots, \theta(a_1)).$$

# Reformulation as an instrument HT

$\widehat{\mathbb{P}}$ is always the unraveling of a reversed $\widehat{\Phi}$ by a reversed instrument $\widehat{\mathcal{J}} := \{\widehat{\Phi}_a\}_a$.

An example is:

$$\widehat{\Phi}_a(X) = J\rho^{-\frac{1}{2}}\Phi^*_{\theta(a)}(\rho^{\frac{1}{2}}JXJ\rho^{\frac{1}{2}})\rho^{-\frac{1}{2}}J$$

with $J : M_d(\mathbb{C}) \to M_d(\mathbb{C})$ an anti unitary involution such that $[J, \rho] = 0$.

Remark that $\widehat{\Phi}$ is the dual of $J\Phi(J \cdot J)J$ w.r.t. the inner product $\langle \cdot, \cdot \rangle_{\frac{1}{2}}$.

# Relative entropy convergence

The entropy production is

$$S_n(\mathbb{P}|\widehat{\mathbb{P}}) := \sum_{a_1,\ldots,a_n} \mathbb{P}(a_1,\ldots,a_n) \log \frac{\mathbb{P}(a_1,\ldots,a_n)}{\widehat{\mathbb{P}}(a_1,\ldots,a_n)} \geq 0.$$

**Theorem (B., Jakšić, Pautrat, Pillet '16)**
*Assume $\Phi$ is irreducible. Then,*

$$Ep := \lim_{n\to\infty} \frac{1}{n} S_n(\mathbb{P}|\widehat{\mathbb{P}})$$

*exists and*

$$Ep = 0 \Leftrightarrow \mathbb{P} = \widehat{\mathbb{P}}.$$

What is the relation between $\mathbb{P} = \widehat{\mathbb{P}}$ and **($\frac{1}{2}$-QDB)**?

**Theorem (Stinespring's dilation theorem '55)**

*If $\Phi$ is a quantum channel, there exists $k \in \mathbb{N}$ and an isometry $V : \mathbb{C}^d \to \mathbb{C}^k \otimes \mathbb{C}^d$ such that*

$$\Phi(X) = V^*(I_k \otimes X)V \quad \forall X \in M_d(\mathbb{C}).$$

*Moreover if $V$ and $W$ are two Stinespring dilations of the same quantum channel $\Phi$, then there exists a unitary matrix $U \in M_k(\mathbb{C})$ such that*

$$W = (U \otimes I_d)\ V.$$

# Informationally complete instruments

**Proposition**

*Given a dilation $V$ of $\Phi$, then for any instrument $\mathcal{J}$ that sums to $\Phi$, there exists a POVM $\{M_a\}_{a\in\mathcal{A}}$ such that for any $a \in \mathcal{A}$,*

$$\Phi_a(X) = V^*(M_a \otimes X)V \quad \forall X \in M_d(\mathbb{C}).$$

**Definition (Informationally complete POVM)**

The POVM $\{M_a\}_{a\in\mathcal{A}}$ is said informationally complete if $\operatorname{linspan}\{M_a\}_{a\in\mathcal{A}} = M_k(\mathbb{C})$.

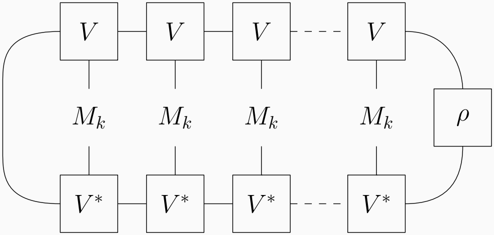
**Definition (Informationally complete instrument)**

The instrument $\mathcal{J}$ is informationally complete if there exists an informationally complete POVM $\{M_a\}_{a\in\mathcal{A}}$ that can generate the instrument $\mathcal{J}$.

# Finitely correlated state (FCS)

If $\mathcal{J}$ is an informationally complete instrument, $\mathbb{P} \equiv \omega$ with $\omega$ a purely generated FCS on $\bigotimes_{n\in\mathbb{Z}} M_k(\mathbb{C})$.

$$\mathbb{P} \equiv \omega =$$




If $A = \sum_{a_1,\ldots,a_n} c_{a_1,\ldots,a_n} M_{a_1} \otimes \cdots \otimes M_{a_n}$, $\omega(A) = \mathbb{E}(X_A)$ with $X_A = \sum_{a_1,\ldots a_n} c_{a_1,\ldots,a_n} \mathbf{1}_{a_1,\ldots,a_n}$.

**Definition (Unitarily implementable involution)**

For a given POVM $\{M_a\}_{a\in\mathcal{A}}$, we say that the local involution $\theta$ is unitarily implementable if there exists a **unitary involutive matrix** $U \in M_k(\mathbb{C})$, such that

$$UM_aU = M_{\theta(a)}.$$

**Theorem (B., Cuneo, Jakšić, Pautrat, Pillet '19)**

*Assume $\Phi$ is primitive. Then, the following are equivalent.*

- $\Phi$ *verifies* **($\frac{1}{2}$-QDB)***;*
- *There exists an informationally complete instrument $\mathcal{J}$ summing to $\Phi$ and a unitarily implementable local involution $\theta$ for an informationally complete POVM generating $\mathcal{J}$ such that $Ep = 0$.*

**Proof:**

$\Rightarrow$ Stinespring's dilation theorem and the duality operation is an involution.
$\Leftarrow$ [Fannes, Nachtergaele, Werner JFA '94] uniqueness of purely generated FCS and the duality operation is an involution. $\square$