Repeated quantum measurements and time arrow hypothesis testing

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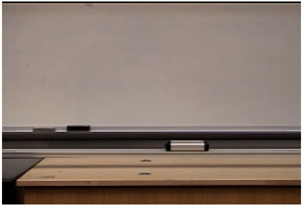
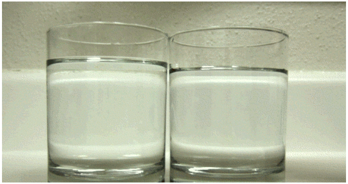
Reversibility and irreversibility



Edited from Wake Forest University Physics Department video.



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Reversibility vs. Irreversibility

Time reversal invariance: $\exists \Theta$ an involution such that,

$$\langle O \rangle_{0 \to t} = \langle \Theta(O) \rangle_{t_f \to t_f - t}.$$

Example: $\Theta(x) = x, \Theta(p) = -p$.

Irreversibility:
Clausius (1850), thermodynamic:

$$Ep := \Delta S \geq 0,$$

Entropy always increases $\Rightarrow$ Thermodynamic time ordering.

Fluctuation relations

Developed in the nineties [Evans, Cohen, Gallavotti, Morris, Crooks ...].

- $\omega = (x_t)_t$: a path in phase space,
- $\sigma_t(\omega)$: entropy production rate random variable ($\mathbb{E}(\sigma_t) = \frac{1}{t} Ep_t$).

If the dynamical system is time reversal invariant, the transient fluctuation relation,

$$\frac{dP_t(\sigma_t = s)}{dP_t(\sigma_t = -s)} = e^{t\,s}$$

holds.
Under Chaotic Hypothesis, for any open set $O \subset \mathbb{R}$

$$\lim_{t\to\infty} \frac{1}{t} \log P(t\sigma_t \in O) = -\inf_{s\in O} I(s),$$

With, $s \mapsto I(s)$ a good rate function such that:

$$I(s) \geq 0, \quad I(s) = 0 \iff s = Ep \quad \text{and} \quad I(s) = I(-s) - s.$$

Projection postulate and irreversibility

von Neumann (1932), 2 rules for quantum mechanics:

1. Projection postulate (PP): $\rho \to \rho' = \frac{P\rho P}{\mathrm{tr}[P\rho']}$ with proba $\mathrm{tr}[P\rho]$ *(Irreversible)*,
2. Unitary evolution: $\rho_t = e^{-iHt}\rho e^{iHt}$ *(Reversible)*.

Projection Postulate Irreversibility $\Rightarrow$ Quantum time ordering.

- Bohm (1951): *"This [quantum] irreversibility greatly resembles that which appears in thermodynamic processes"*.
- Landau-Lifschitz (1978): 2^{nd} law macroscopic expression of PP?

Criticism: Two state–vector formalism

Aharonov, Bergmann and Lebowitz (1964): *"This time asymmetry is actually related to the manner in which statistical ensembles are constructed"*

$$p(a \text{ then } j_1, j_2, \ldots, j_n; b) \neq p(b \text{ then } j_n, j_{n-1}, \ldots, j_1; a)$$

but,

$$p(j_1, j_2, \ldots, j_n; b|a) := |\langle b|j_n\rangle|^2 |\langle j_n|j_{n-1}\rangle|^2 \cdots |\langle j_2|j_1\rangle|^2 |\langle j_1|a\rangle|^2 = p(j_n, \ldots, j_2, j_1; a|b).$$

Conditioning on initial and final states, restores time reversal invariance (TRI) at the level of the measurement statistics.

Full Counting Statistics and Fluctuation Relations

Kurchan (2000):

$$H_i = \sum_{\epsilon} \epsilon |\epsilon\rangle\langle\epsilon|, \quad H_f = \sum_{\epsilon'} \epsilon' |\epsilon'\rangle\langle\epsilon'|$$

Work distribution:

$$P_t(W) = \sum_{W=\epsilon'_f-\epsilon_i} p(\epsilon_i \text{ then } \epsilon'_f) = \sum_{W=\epsilon'_f-\epsilon'_i} |\langle\epsilon'_f|U\epsilon_i\rangle|^2 \, \frac{e^{-\beta\epsilon_i}}{Z}.$$

$\rho \propto e^{-\beta H_i} \to |i\rangle\langle i|$ $\qquad U \qquad$ $U|i\rangle \to |f\rangle$

$0 \vdash\!\!\longrightarrow t$

$H_i \to \epsilon_i$ $\qquad W = \epsilon'_f - \epsilon_i \qquad$ $H_f \to \epsilon_f$

Time Reversal Invariance $\Rightarrow$ Crooks Fluctuation Relation: If,

$$\exists\Theta, \text{ s.t. } \Theta(\Theta(X)) = X, \quad \Theta(i\mathbb{1}) = -i\mathbb{1}, \quad \Theta(U) = U^*, \quad \Theta(\rho_i) = \rho_f$$

then,

$$\frac{dP_t(W = w)}{dP_t(W = -w)} = e^{\beta w}.$$

Entropic fluctuation relation

Entropy production defined as a two time measurement of the system state:

1. Mesure $-\ln \rho$: value s_i,
2. Evolve with $U := e^{-it(H+V)}$,
3. Measure $-\ln \rho$: value s_f.
4. **Entropy production:** $t\sigma := s_f - s_i$.

If time reversal invariance is verified:

$$\exists \Theta, \text{ s.t. } \Theta(i\mathbb{1}) = -i\mathbb{1}, \quad \Theta(\rho) = \rho, \quad \Theta(H) = H, \quad \Theta(V) = V.$$

Then the statistic of σ verifies:

$$S(\rho_t|\rho) = t\int_{\mathbb{R}} \sigma dP_t(\sigma) \quad \text{and} \quad \frac{dP_t(\sigma = s)}{dP_t(\sigma = -s)} = e^{t\,s}.$$

$\Rightarrow$ Positive entropy production is exponentially more likely.

Issue: Two time projective measurement of a non local quantity.
$\Rightarrow$ experimental propositions of measurement of the FCS using an auxiliary qbit interacting locally: Campisi, M. *et al.* New J. Phys. **15** (2013); Dorner, R. *et al.* PRL **110** (2013); Goold, J.*et al.* PRE **90** (2014); Mazzola, L. *et al.* PRL **110** (2013); Roncaglia, A.J. *et al.* PRL **113** (2014).

Entropy production of repeated measurements

Quantify: When can one choose the right movie order ? If it is possible how does the probability of error decays ?

Why study repeated indirect measurements ?

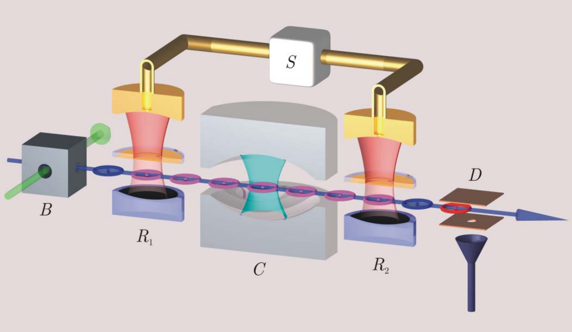
- Experimentally relevant (Cavity QED, Interferometry . . .),
- “Every day experience”,
- *Because we can have results.*

Outline

1. Repeated measurement model,
2. Entropy production and distinguishability between forward and backward,
3. Rényi relative entropy regularity, Fluctuation relations,
4. Hypothesis testing and error exponents.

A canonical experiment

S. Haroche group experiment:

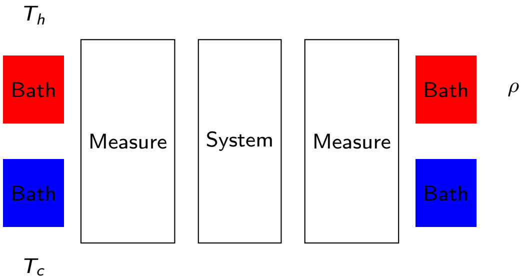


j	110111111111001110110111 1
i	ddcbccabcdaadaabadddbadbc
j	0101001101010101101011111
i	dababbaacbccdadccdcbaaacc

j	0001000110110000001010110
i	ddcaddabbccdccbcdaabbccab
j	0001010100000100011101101
i	bcdaddaabbbbdbdcdccadaada

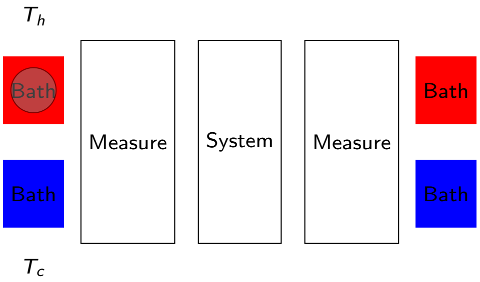
Image: LKB ENS

Repeated indirect measurements, a two heat baths example



Hilbert space: $\mathcal{H} := \mathbb{C}^2_{sys.} \otimes \mathbb{C}^2_{hot} \otimes \mathbb{C}^2_{cold}$.

Repeated indirect measurements, a two heat baths example



$\rho \otimes \beta_h$, prob. $\frac{1}{2}$

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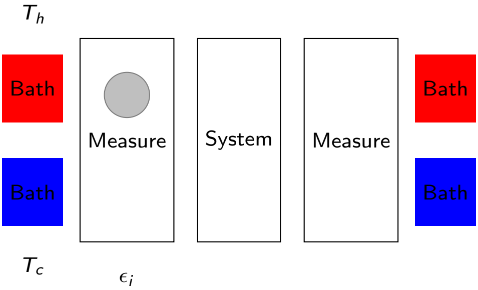
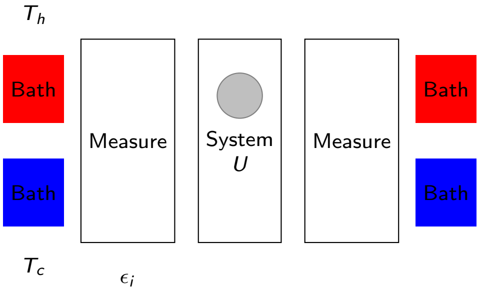


$\rho \otimes |i\rangle\langle i|$, prob. $\langle i|\beta_h i\rangle$

Repeated indirect measurements, a two heat baths example



$U(\rho \otimes |i\rangle\langle i|)U^*$

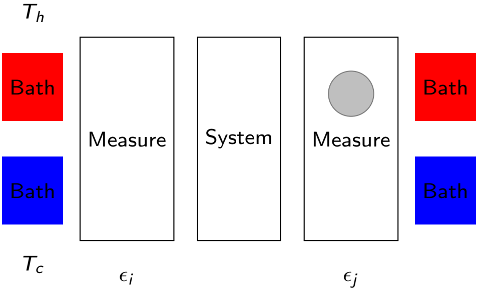
Dipolar, RWA:

$$U = \exp(i\tau H_{RWA})$$

$$H_{RWA} = \omega\sigma_z \otimes I + \omega I \otimes \sigma_z + \lambda\sigma_+ \otimes \sigma_- + h.c.$$

$$H_{full} = \omega\sigma_z \otimes I + \omega I \otimes \sigma_z + \lambda\sigma_x \otimes \sigma_x$$

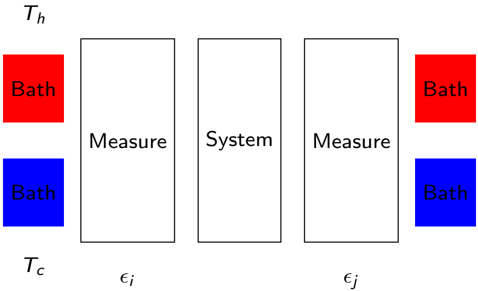
Repeated indirect measurements, a two heat baths example



$$\frac{U_{ji}\rho U_{ij}^*}{\mathrm{tr}[U_{ji}\rho U_{ij}^*]} \otimes |j\rangle\langle j|,$$

prob. $\mathrm{tr}[U_{ji}\rho U_{ij}^*] \times \langle i|\beta_h i\rangle$

Repeated indirect measurements, a two heat baths example



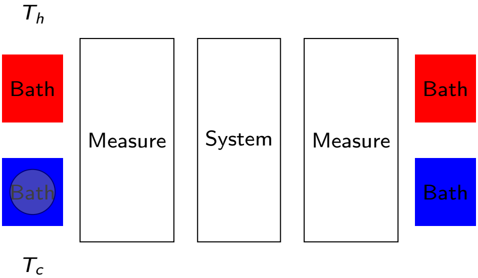
$$\rho(h;j,i) := \frac{V_{h;j,i}\rho V^*_{h;j,i}}{\mathrm{tr}[V^*_{h;j,i}V_{h;j,i}\rho]},$$

prob. $\mathrm{tr}[V^*_{h;j,i}V_{h;j,i}\rho]$.

$$V_{h;j,i} = U_{ji}\sqrt{\langle i|\beta_h i\rangle},$$

$$\Delta E = \epsilon_j - \epsilon_i.$$

Repeated indirect measurements, a two heat baths example



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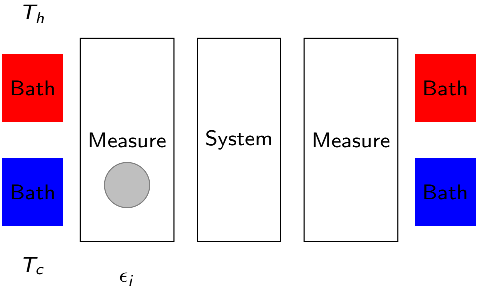

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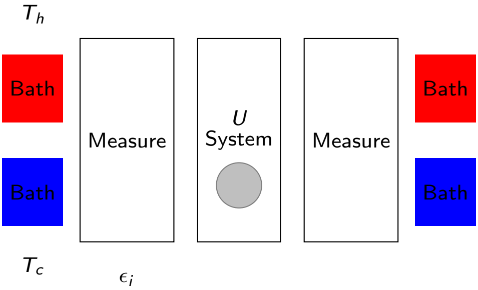

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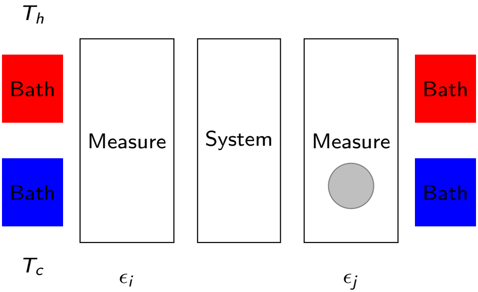

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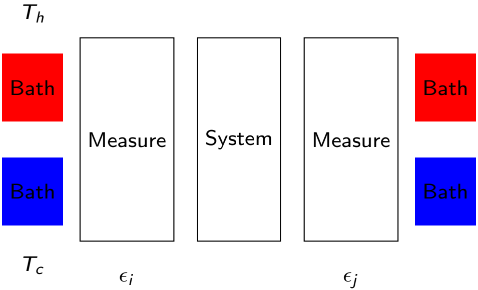

$\rho(c;j,i) := \frac{V_{c;j,i}\rho V^*_{c;j,i}}{\mathrm{tr}[V^*_{c;j,i}V_{c;j,i}\rho]}$,

prob. $\mathrm{tr}[V^*_{c;j,i}V_{c;j,i}\rho]$.

$$V_{c;j,i} = U_{ji}\sqrt{\langle i|\beta_c i\rangle},$$

$$\Delta E = \epsilon_j - \epsilon_i.$$

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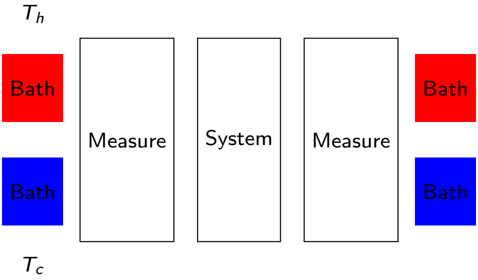


$\rho(c;j,i) := \frac{V_{c;j,i}\rho V_{c;j,i}^*}{\mathrm{tr}[V_{c;j,i}^* V_{c;j,i}\rho]}$,

prob. $\mathrm{tr}[V_{c;j,i}^* V_{c;j,i}\rho]$.

Measurement result sequence: $((b_1; i_1, j_i), \ldots, (b_t; i_t, j_t)) = (k_1, \ldots, k_t)$ with probability

$$\mathbb{P}(k_1, \ldots, k_t) = \mathrm{tr}[V_{k_t} \cdots V_{k_1} \rho V_{k_1}^* \cdots V_{k_t}].$$

Remark: Two time measurement process studied by Crooks[PRA '08] and Horowitz, Parrondo[NJP '13]

Quantum instruments

Definition (Instruments)

Let

$$\mathcal{J} := \{\Phi_k : M_d(\mathbb{C}) \to M_d(\mathbb{C})\}_{k=1,\dots,\ell}$$

be a familly of completely positive (CP) maps such that the CP map

$$\Phi := \sum_{k=1}^{\ell} \Phi_k$$

is unital (CPU). Then $\mathcal{J}$ is called an instrument.
If moreover the Kraus rank of Φ_k is 1 for any $k = 1, \dots, \ell$, the instrument is called *perfect*.

Definition (Unraveling)

Let $\rho \in M_d^{+,1}(\mathbb{C})$ be a state on $\mathbb{C}^d$, then the probability measure $\mathbb{P}$ on $\Omega = \{1, \dots, \ell\}^{\mathbb{N}}$ defined by the marginals,

$$\mathbb{P}_t(k_1, \dots, k_t) = \mathrm{tr}(\rho \Phi_{k_1} \circ \cdots \circ \Phi_{k_t}(I_d))$$

is called an unraveling of the CPU map Φ.

The dynamical system picture

ω_0	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	$\cdots$	ω_t	$\cdots$

$\omega_t = 1, \ldots, \ell$.

- States of the dynamical system: $\{(c; i, j), (h; i, j)\}_{i,j} \equiv \{1, \ldots, \ell\}$
- "Trajectory" space: $\Omega = \{1, \ldots, \ell\}^{\mathbb{N}}$, time: $t \in \mathbb{N}$.
  Finite time trajectories:

$$\Omega_t := \{1, \ldots, \ell\}^t, \quad \Omega_{\text{fin}} = \cup_{t \in \mathbb{N}} \Omega_t.$$

- Measure on the trajectories: $\mathbb{P}$,
- Time shift: $f \circ \phi^t(\omega_0, \omega_1, \ldots) = f(\omega_t, \omega_{t+1}, \ldots)$ for all trajectory $\omega \in \Omega$.

Remark: Can also be seen as a classical spin ℓ chain with the configurations probabilities given by $\mathbb{P}$ or a finitely correlated state over a commutative algebra $\mathfrak{C}$.

Dynamical system properties

Ergodic property:

- If ρ is the unique invariant state of Φ^*, then $(\Omega, \mathbb{P}, \phi)$ is ergodic.

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}(g\ f \circ \phi^t) = \mathbb{E}(g)\mathbb{E}(f).$$

From now on, we assume ρ is the unique invariant state of Φ^*.

Upper Bernoulli property:

- $\exists C > 0$ such that, for any finite sequence $k_1, \dots, k_s, k_{s+1}, \dots, k_t$,

$$\mathbb{P}(k_1, \dots, k_s, k_{s+1}, \dots, k_t) \le C\mathbb{P}(k_1, \dots, k_s)\mathbb{P}(k_{s+1}, \dots, k_t).$$

Time reversal

Let $\theta : \{1, \ldots, \ell\} \to \{1, \ldots, \ell\}$ be an involution (*i.e.* $\theta(\theta(k)) = k$).
A time reversal of the measurement results is then:

$$\Theta(k_1, \ldots, k_t) := (\theta(k_t), \ldots, \theta(k_1)).$$

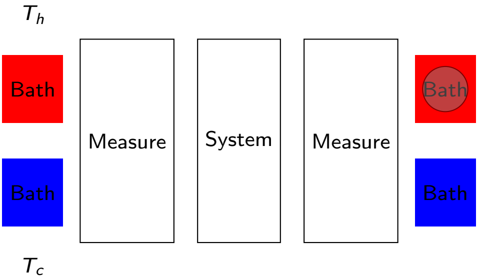
The time reversed probability measure over Ω is:

$$\widehat{\mathbb{P}}(k_1, \ldots, k_t) = \mathbb{P}(\theta(k_t), \ldots, \theta(k_1)).$$

$\widehat{\mathbb{P}}$ is the unraveling of $\widehat{\Phi}$ by the instrument $\widehat{\mathcal{J}} := \{\widehat{\Phi}_k\}_k$ with

$$\widehat{\Phi}_k(X) = \rho^{-\frac{1}{2}} \Phi^*_{\theta(k)}(\rho^{\frac{1}{2}} X \rho^{\frac{1}{2}}) \rho^{-\frac{1}{2}}.$$

Time reversal of the two baths example



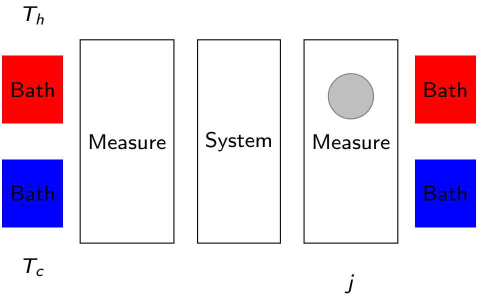
$\theta(b; i, j) = (b; j, i)$

$$((c, +\Delta E), (h, -\Delta E), \ldots, (h, 0))$$

reversed is

$$((c, -\Delta E), (h, +\Delta E), \ldots, (h, 0)).$$

Time reversal of the two baths example



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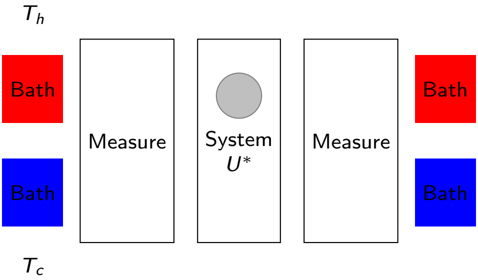

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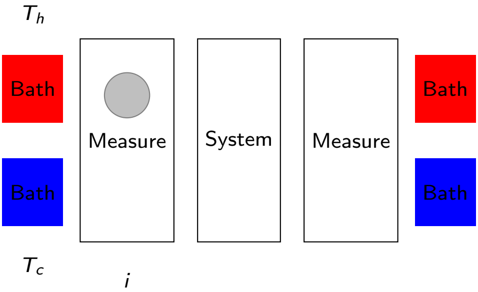


$\theta(b; i, j) = (b; j, i)$

$$((c, +\Delta E), (h, -\Delta E), \ldots, (h, 0))$$

reversed is

$$((c, -\Delta E), (h, +\Delta E), \ldots, (h, 0)).$$

Comparing $\widehat{\mathbb{P}}$ and $\mathbb{P}$

Assume non finite time distinguishability: $\mathbb{P}_t(A) > 0 \Leftrightarrow \widehat{\mathbb{P}}_t(A) > 0$ for all $t \in \mathbb{N}$.

We study the entropy production,

- In mean:
$$S(\mathbb{P}_t|\widehat{\mathbb{P}}_t) := \sum_{\omega_t} \mathbb{P}_t(\omega_t) \log[\mathbb{P}_t(\omega_t)/\widehat{\mathbb{P}}_t(\omega_t)] \geq 0.$$
- Random variable:
$$\sigma_t = \frac{1}{t} \log[\mathbb{P}_t(\omega_t)/\widehat{\mathbb{P}}_t(\omega_t)].$$

Since the time reversal is an involution:

$$S(\mathbb{P}_t|\widehat{\mathbb{P}}_t) = \sum_{\omega_t \in \Omega_t} \mathbb{P}_t(\omega_t) \log[\mathbb{P}_t(\omega_t)/\widehat{\mathbb{P}}_t(\omega_t)] = S(\widehat{\mathbb{P}}_t|\mathbb{P}_t).$$

Two sub additive convergence results

Lemma (Fekete)

Let $(a_t)_{t\geq 1}$ be a sequence of real numbers such that for a $c \in \mathbb{R}$ and all $s, t \in \mathbb{N}$,

$$a_{t+s} \leq a_t + a_s + c.$$

Then

$$\lim_{t\to\infty} \frac{1}{t} a_t = \inf_{t\geq 1} \frac{a_t + c}{t}.$$

Theorem (Kingman)

Let $X_t : \Omega \to \mathbb{R}$ be a sequence of random variables such that $\mathbb{E}(|X_t|) < \infty$. Assume $\exists C \in \mathbb{R}$ such that for all $t, s \in \mathbb{N}$,

$$X_{t+s}(\omega) \leq X_t(\omega) + X_s \circ \phi^t(\omega) + C$$

with $\mathbb{P}$ probability 1. Then the limit

$$x(\omega) := \lim_{t\to\infty} \frac{1}{t} X_t(\omega)$$

exists with probability 1 and is ϕ invariant. Moreover

$$\lim_{t\to\infty} \frac{1}{t} \mathbb{E}(X_t) = \mathbb{E}(x).$$

Entropy production

Theorem (B., Jaksic, Pautrat, Pillet '16)

$$Ep := \lim_{t \to \infty} \frac{1}{t} S(\mathbb{P}_t | \widehat{\mathbb{P}}_t)$$

exists. Assume moreover that $\mathbb{P}$ *is ergodic. Then,*

$$\sigma := \lim_{t \to \infty} \sigma_t = \mathbb{E}(\sigma) = Ep. \quad \mathbb{P} - \textit{almost surely.}$$

Moreover,

$$Ep = 0 \Leftrightarrow \mathbb{P} = \widehat{\mathbb{P}} \quad \textit{and} \quad Ep > 0 \Leftrightarrow \mathbb{P}(\sigma > 0) = 1 \textit{ and } \widehat{\mathbb{P}}(\sigma > 0) = 0.$$

Remark

With $\widehat{\mathbb{P}}$ *probability* 1,

$$\lim_{t \to \infty} \sigma_t = -Ep.$$

Entropy production

The asymptotic entropy production random variable distinguishes between $\mathbb{P}$ and $\widehat{\mathbb{P}}$. Given an observed "trajectory" $\omega \in \Omega$,

- Either $\sigma \geq 0$ and the arrow goes forward (i.e. $\mathbb{P}$ is the underlying measure),
- Or $\sigma \leq 0$ and the arrow goes backward (i.e. $\widehat{\mathbb{P}}$ is the underlying measure).

Remark

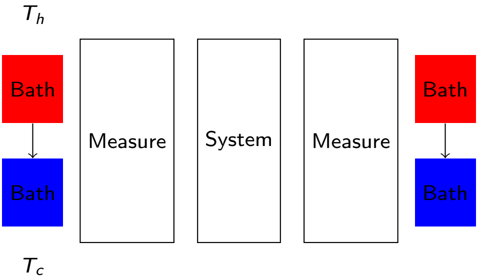
$$Ep = 0 \sim \textit{Detailed balance condition.}$$

With Detailed balance condition:

$$\Phi \equiv \widehat{\Phi}.$$

$(\leftarrow)$ *can be proved for a family of appropriate measurements.*
$(\rightarrow)$*? from the theory of finitely correlated states [Fannes, Nachtergaele, Werner CMP '92].*

Entropy production



$$\sigma_t \simeq \frac{1}{t}(\frac{1}{T_h}\Delta Q_h + \frac{1}{T_c}\Delta Q_c).$$

Since $J_Q := \lim_{t\to\infty} \frac{1}{t}\Delta Q_c = \lim_{t\to\infty} -\frac{1}{t}\Delta Q_h$ with probability 1,

$$Ep = \frac{T_h - T_c}{T_c T_h} J_Q > 0 \quad \Rightarrow \quad \text{sign}(J_Q) = \text{sign}(T_h - T_c).$$

Since $J_Q = \Delta E \frac{1}{2}(\mathbb{P}(c; +\Delta E) - \mathbb{P}(c; -\Delta E))$,

$$Ep = 0 \Leftrightarrow T_c = T_h.$$

Remark: This is not true for full dipolar interaction where $Ep > 0$ even if $T_h = T_c$.

Beyond the law of large numbers: Rényi relative entropy.

Cumulant generating function of $-t\sigma_t$:

$$e_t(\alpha) := \log \sum_{\omega_t \in \Omega_t} \mathbb{P}_t(\omega_t)^{(1-\alpha)} \widehat{\mathbb{P}}_t(\omega_t)^{\alpha} = S_\alpha(\mathbb{P}_t|\widehat{\mathbb{P}}_t).$$

Since $\sum_{\omega_t} f(\omega_t) = \sum_{\hat{\omega}_t} f(\hat{\omega}_t)$,

$$e_t(\alpha) = S_\alpha(\mathbb{P}_t|\widehat{\mathbb{P}}_t) = S_{1-\alpha}(\widehat{\mathbb{P}}_t|\mathbb{P}_t) = S_{1-\alpha}(\mathbb{P}_t|\widehat{\mathbb{P}}_t) = e_t(1-\alpha).$$

Hence, the transient fluctuation relation holds:

$$\frac{\mathbb{P}_t(\sigma_t = s)}{\mathbb{P}_t(\sigma_t = -s)} = e^{t\,s}.$$

Theorem (B.,Jaksic, Pautrat, Pillet '16)

$\forall \alpha \in [0,1]$,

$$e(\alpha) := \lim_{t\to\infty} \frac{1}{t} e_t(\alpha)$$

exists, is continuous, convex, satisfies $e(0) = e(1) = 0$ *and*

$$e(\alpha) = e(1-\alpha),$$

$$\partial_\alpha^+ e(\alpha)|_{\alpha=0} = -\partial_\alpha^- e(\alpha)|_{\alpha=1} = -Ep.$$

Rényi relative entropy as an entropic pressure

Rényi entropy can be obtained through a variational principle.

$$\frac{1}{t}e_t(\alpha) = \frac{1}{t}\max_{\mathbb{Q}_t}(\mathbb{E}_{\mathbb{Q}_t}(\log \mathbb{P}_t) - \alpha\mathbb{E}_{\mathbb{Q}_t}(\sigma_t) + S(\mathbb{Q}_t)).$$

Thermodynamic equivalent: Canonical Gibbs distribution maximises the free energy.

$$F_L \sim e_t(\alpha), \quad S_L \sim S(\mathbb{Q}_t) \quad \text{and} \quad \beta E_L \sim \alpha\mathbb{E}_{\mathbb{Q}_t}(\sigma_t) - \mathbb{E}_{\mathbb{Q}_t}(\log \mathbb{P}_t).$$

Since $E_{t+s} \leq C + E_t + E_s \Rightarrow$ sub additive thermodynamic formalism[1] $\Rightarrow$ regularity of $e(\alpha)$.

Let $\mathcal{P}_\phi$ be the set of ϕ invariant probability measures over Ω.
For all $\alpha \in [0,1]$ there exists $\mathbb{Q} \mapsto f_\alpha(\mathbb{Q})$ affine and upper semicontinuous such that:

$$e(\alpha) = \sup_{\mathbb{Q}\in\mathcal{P}_\phi} f_\alpha(\mathbb{Q})$$

Let $\mathcal{P}_{eq}(\alpha)$ be the set of probability measures for which the supremum is reached.
If $\mathcal{P}_{eq}(\alpha)$ is a singleton, then $\alpha \mapsto e(\alpha)$ is differentiable on $]0,1[$.

[1] [Barreira '10, Feng '09]

Differentiability of $e(\alpha)$

Assumption (C): (Weaker than lower Bernoulli) There exists τ and $C' > 0$ such that for all s, t, $\omega_t \in \Omega_t, \nu_s \in \Omega_s$, there exists $\xi_u \in \Omega_u$ with $u \leq \tau$ such that

$$\mathbb{P}(\omega_t, \xi_u, \nu_s)\widehat{\mathbb{P}}(\omega_t, \xi_u, \nu_s) \geq C'\mathbb{P}(\omega_t)\mathbb{P}(\nu_s)\widehat{\mathbb{P}}(\omega_t)\widehat{\mathbb{P}}(\nu_s).$$

Theorem (B., Jaksic, Pautrat, Pillet, '16)

If Assumption (C) holds, $\alpha \mapsto e(\alpha)$ *is differentiable on* $]0, 1[$.

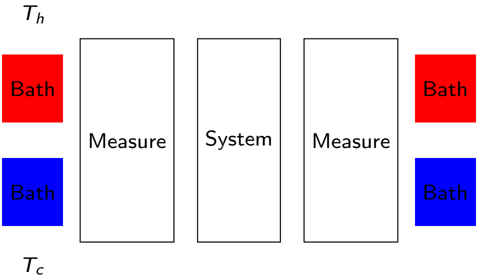
Assumption (D): (Quasi Bernoulli) There exists $C > 0$ such that for all s, t, $\omega_t \in \Omega_t, \nu_s \in \Omega_s$,

$$C^{-1}\mathbb{P}(\omega_t)\mathbb{P}(\nu_s) \leq \mathbb{P}(\omega_t, \nu_s) \leq C\mathbb{P}(\omega_t)\mathbb{P}(\nu_s).$$

Theorem (B., Jaksic, Pautrat, Pillet '16)

If Assumption (D) holds, $\alpha \mapsto e(\alpha)$ *exists and is differentiable on* $\mathbb{R}$.

Rényi entropy and heat cumulant generating function.

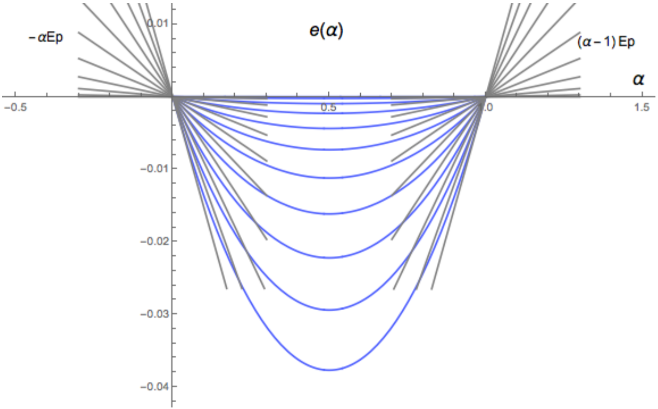


$e(\alpha)$ is the limit cumulant generating function of

$$-\sigma_t \simeq \frac{1}{t} \frac{T_c - T_h}{T_c T_h} \Delta Q_c .$$

It can be explicitly computed using spectral techniques on CP maps.

$e(\alpha)$ for different $T_h - T_c$



$e(1/2)$ decreases when $T_h - T_c$ increases.

Fluctuation relation

The entropy production random variable verifies a local large deviation principle.

$$I(s) := \sup_{\alpha \in \mathbb{R}} (\alpha s - \overline{e}(\alpha)).$$

From the symmetry $e(\alpha) = e(1-\alpha)$, this rate function is such that

$$I(-s) - I(s) = s \quad \text{and} \quad I(Ep) = 0.$$

Theorem

If Assumption (C) holds, for any $s \in]-Ep, Ep[$,

$$\lim_{\epsilon \downarrow 0} \lim_{t \to \infty} \frac{1}{t} \log \mathbb{P}_t(|\sigma_t - s| < \epsilon) = -I(s)$$

$$\lim_{\epsilon \downarrow 0} \lim_{t \to \infty} \frac{1}{t} \log \mathbb{P}_t(|\sigma_t + s| < \epsilon) = -I(-s) = -(I(s) + s)$$

If Assumption (D) holds, then both previous limit hold for any $s \in \mathbb{R}$.

Hypothesis testing of the arrow of time

Aim: Evaluation of the error one can make when guessing the arrow of time.

H_0 The observed quantum measurements are described by $(\mathcal{J}, \rho)$.

H_1 The observed quantum measurements are described by $(\widehat{\mathcal{J}}, \rho)$.

For each time t let $\mathcal{T}_t$ be an event whose realisation implies we decide "H_0 is true".

$$\text{Example: } \underline{\mathcal{T}}_t = \{\omega_t \in \Omega_t | \sigma_t > 0\}.$$

Then,

- $\mathbb{P}_t(\mathcal{T}_t^c)$ is the probability to reject H_0 when it is true (Type I error).
- $\widehat{\mathbb{P}}_t(\mathcal{T}_t)$ is the probability to accept H_0 when H_1 is true (Type II error).

Stein's error exponents

Stein error exponent for $\epsilon \in]0,1[$:

$$s_t(\epsilon) := \min_{\mathcal{T}_t}\{\widehat{\mathbb{P}}_t(\mathcal{T}_t) | \mathcal{T}_t \subset \Omega_t \text{ and } \mathbb{P}_t(\mathcal{T}_t^c) \leq \epsilon\}.$$

"$s_t(\epsilon)$ is the minimal error of type II while we control the error of type I."

Theorem (adapted from Jaksic, Ogata, Pillet, Seiringer '12)

Assume Φ is irreducible. Then, for all $\epsilon \in]0,1[$,

$$\lim_{t\to\infty} \frac{1}{t} \log s_t(\epsilon) = -Ep.$$

The entropy production corresponds to the exponential decreasing rate of the error of type II given any control on the error of type I.

Hoeffding error exponents

These exponents are similar to the Stein one, with a tighter control on the type I error.

$$\overline{h}(s) := \inf_{\mathcal{T}_t}\{\limsup_{t\to\infty} \frac{1}{t} \log \widehat{\mathbb{P}}_t(\mathcal{T}_t) | \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}_t(\mathcal{T}_t^c) < -s\}$$

$$\underline{h}(s) := \inf_{\mathcal{T}_t}\{\liminf_{t\to\infty} \frac{1}{t} \log \widehat{\mathbb{P}}_t(\mathcal{T}_t) | \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}_t(\mathcal{T}_t^c) < -s\}$$

$$h(s) := \inf_{\mathcal{T}_t}\{\lim_{t\to\infty} \frac{1}{t} \log \widehat{\mathbb{P}}_t(\mathcal{T}_t) | \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}_t(\mathcal{T}_t^c) < -s\}.$$

For $s \geq 0$, set

$$\Psi(s) = -\sup_{\alpha\in[0,1[} \frac{-s\alpha - e(\alpha)}{1-\alpha}.$$

Theorem (adapted from Jaksic, Ogata, Pillet, Seiringer '12)

Suppose Φ is irreducible and Assumption (C) holds. Then for $s \geq 0$,

$$\underline{h}(s) = \overline{h}(s) = h(s) = \Psi(s).$$

Chernoff exponents

Assume a priori equiprobability for both hypothesis H_0 and H_1. Take the test,

$$\mathcal{T}_t = \{\omega_t \in \Omega_t | \sigma_t > 0\}.$$

Then, the total probability of error is:

$$c_t := \frac{1}{2}\mathbb{P}_t(\mathcal{T}_t^c) + \frac{1}{2}\widehat{\mathbb{P}}_t(\mathcal{T}_t) = \frac{1}{2}(1 - \|\mathbb{P}_t - \widehat{\mathbb{P}}_t\|_{TV}).$$

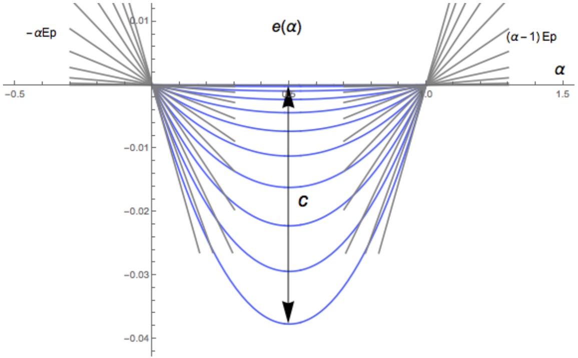
Chernoff exponents are:

$$\overline{c} := \limsup_{t\to\infty} \frac{1}{t} \log c_t \quad \text{and} \quad \underline{c} := \liminf_{t\to\infty} \frac{1}{t} \log c_t.$$

Theorem (adapted from Jaksic, Ogata, Pillet, Seiringer '12)

- $\overline{c} \leq e(\frac{1}{2})$ *and* $\underline{c} \geq e(\frac{1}{2}) - \frac{1}{2}\partial^+ e(\frac{1}{2})$.
  Particularly $Ep > 0 \Rightarrow \overline{c} < 0$.
- *If Assumption (C) holds,* $\overline{c} = \underline{c} = e(\frac{1}{2})$.

$e(\alpha)$ and error exponents



Open questions

- General algebraic characterisation of Assumption (C),
- Φ irreducible and $\mathbb{P}_t \sim \widehat{\mathbb{P}}_t$ $\forall t$ such that (C) does not hold,
- Irregularities outside $]0,1[$ and higher order Stein's exponents (Example where $\frac{\sigma_t - Ep}{\sqrt{t}} \to \gamma(\mathcal{N}(0,1) - |\mathcal{N}(0,1)|)$),
- Other conditions on Φ such that it verifies detailed balance,
- Continuous time version,
- Entangled probes,
- Time reversal of the underlying Markov chain on the system.