Master level internship

Large deviation principle for quantum trajectories

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Quantum trajectories are Markov chains on projective spaces modeling the evolution of a quantum system under repeated (indirect) measurements. For x in the unit sphere of $\mathbb{C}^d$, let $\hat{x} = \{\lambda x : \lambda \in \mathbb{C}\}$. The projective space of $\mathbb{C}^d$ is $\mathrm{P}(\mathbb{C}^d) = \{\hat{x} : x \in \mathbb{C}^d \setminus \{0\}\}$. Let $\pi_{\hat{x}}$ be the orthogonal projector onto $\hat{x}$. Given a family of matrices $\{v_i\}_{i\in I} \subset M_d(\mathbb{C})$ such that $\sum_i v_i^* v_i = \mathrm{id}_{\mathbb{C}^d}$, a quantum trajectory is a Markov chain $(\hat{x}_n)_{n\in\mathbb{N}}$ taking value in $\mathrm{P}(\mathbb{C}^d)$ and defined by

$$\pi_{\hat{x}_{n+1}} = \frac{v_i \pi_{\hat{x}_n} v_i^*}{\mathrm{tr}(v_i \pi_{\hat{x}_n} v_i^*)}, \quad \text{with probability } \mathrm{tr}(v_i \pi_{\hat{x}_n} v_i^*).$$

These processes do not fit the usual ergodic assumptions leading to standard limit theorems. In particular, it has only recently been proved that, under natural assumptions, they accept a unique invariant measure – see [BFPP]. The geometric convergence proved in the same reference has since been refined to prove several limit theorems (law of large numbers, central limit theorem, law of the iterated logarithm, moderate deviation principle, Berry-Esseen bound, Restricted large deviation principle) using Poisson equation methods [BFP] or deformations of the transition kernel and a spectral gap [BHP].

The main limit theorem remaining to prove is the large deviation principle. Namely that the probability that the empirical measure of $(\hat{x}_n)_{n\in\mathbb{N}}$ deviates from the unique invariant measure decays exponentially fast with n with a rate given by an appropriate rate function – see [DZ]. It has been proved to hold for a simple but relevant example of quantum trajectory in [BCPP]. In this example the authors use a Ruelle-Landford function method to prove the existence of a large deviation principle. This approach was inspired by a similar one used to prove a large deviation principle for dynamical systems verifying some approximate decoupling assumptions – see [CJPS]. In [BCPP], the obtained rate function is not convex, which proves that the method based on Gartner-Ellis theorem used in [BHP] to obtain a restricted large deviation principle cannot be improved to obtain a full large deviation principle.

The goal of this internship is to understand the proof of [BCPP] in order to generalize it to quantum trajectories whose invariant measure has finite support. This internship can be continued into a Ph. D. thesis whose goal would be to prove a full large deviation principle for quantum trajectories verifying the assumptions of [BFPP]. The keys to unlock the proposed problems are expected to be found in a fine understanding of the topology of the problem, some clever probabilistic and combinatorial arguments and a deep understanding of the structure of quantum trajectories.

Beyond the application to quantum trajectories, we expect this project will unlock new paths to prove large deviation principles for other non standard Markov chains like iterated function systems appearing in multi-fractal analysis.

References

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