# MASTER LEVEL INTERNSHIP PROPOSAL: ON A CLASS OF MEASURES ON PROJECTIVE SPACES

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We first recall the definition of a projective space. Let $\sim$ be an equivalence relation on $\mathbb{C}^k \setminus \{0\}$ defined by $x \sim y$ iff there exists $\lambda \in \mathbb{C} \setminus \{0\}$ such that $x = \lambda y$. Then the projective space $\mathrm{P}(\mathbb{C}^k)$ is the set of equivalent classes of $\sim$. We denote $\hat{x} \in \mathrm{P}(\mathbb{C}^k)$ the equivalent class of $x \in \mathbb{C}^k \setminus \{0\}$ and given $\hat{x} \in \mathrm{P}(\mathbb{C}^k)$, $x \in \mathbb{C}^k \setminus \{0\}$ is an arbitrary $\ell^2$-norm one element of $\hat{x}$. To fix ideas, $\mathrm{P}(\mathbb{C}^k)$ is isometric to $S^{k-1}/U(1)$ where $S^{k-1}$ is the $k$-dimensional complex $\ell^2$-sphere.

A point $\hat{x}$ in the projective space $\mathrm{P}(\mathbb{C}^k)$ describes the state of quantum system. Though, in experiments only the covariance $\rho = \mathbb{E}(\pi_{\hat{x}})$, where $\pi_{\hat{x}}$ is the orthogonal projector onto $\hat{x}$, is known. Hence a natural (no additional information) probability measure that could describe the system state is given by the probability measure maximizing the entropy

$$h(\nu) = -\int_{\mathrm{P}(\mathbb{C}^k)} \log\left(\frac{d\nu}{d\mu}(\hat{x})\right) d\mu(\hat{x})$$

conditioned on $\int \pi_{\hat{x}} d\nu(\hat{x}) = \rho$ where $\mu$ is the uniform measure on $\mathrm{P}(\mathbb{C}^k)$. These measures have been scarcely studied and become more relevant as quantum optics experiments allow for the "real-time" tracking of individual systems. They may also prove pertinent as tools for data generation in machine learning.

In this internship we propose to first derive more explicit expressions for these maximal entropy measure. We expect useful formulas depending only on the spectrum of $\rho$ could be found using symmetries and Harish-Chandra-Itzykson-Zuber integral formula. From that point the main question is the existence of a high dimension ($k \to \infty$) weak limit for $\nu$ provided the covariance $\rho$ converges in a suitable sense to a traceclass operator.

Depending on the student taste this internship could also focus on other measures $\nu$. For example invariant measures of specific Markov chains (see [1]). Beyond the high dimension limit, we expect a concentration of measure phenomenon may appear. The proof of such concentration is highly relevant in quantum statistical mechanics.

This work could eventually be continued during a Ph.D. Thesis in probability.

## References

[1] T. Benoist, M. Fraas, Y. Pautrat and C. Pellegrini, *Invariant measure for quantum trajectories*, Probab. Theory Relat. Fields **174** (2019) 307–334

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